Astrophysics

Summary

`mantoinette`, January 30, 2023

1 Introduction

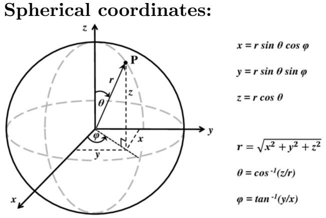
1.1 Coming of age in the Milky Way

- **300 Ptolemy:** A precise mathematical model to predict the apparent positions of the sun, moon, planets in the sky via Earth-centered model.
- **1605-1620 Kepler:** Three laws of planetary motion.
- **1687 Newton:** Action at a distance: $F = \frac{GMm}{r^2}$.
- **1838:** Parallax, Star clusters, Nebulae.
- **1900:** Beginning of modern Astrophysics. Light detectors on large telescopes. Researching atmospheres of the sun and stars.
- **1915-1917:** General relativity and cosmology.
- **1930-1950:** Theories of stellar structure, Nuclear physics (Fusion as the energy source in the sun), Theory of stellar evolution.
- **1960-1970:** Opening of the electromagnetic spectrum discovered Neutron stars, Black holes and Quasars.
- **1970:** non-photon astronomy: neutrinos.
- **1967:** Big Bang cosmology.
- **1980:** Dark Matter and Galaxy formation and evolution.
- **1990:** Exoplanets (4000+).
- **2000:** Fluctuations in the CMB and Dark energy.

1.2 Astrophysical units and coordinates

	S.I.	c.g.s.	Often used in astrophysics
time	s	s	**yr** = 3.154x10^{7} s, Myr, Gyr
length	m	cm	**AU** = 1.496x10^{11} m, **pc** = 3.086x10^{16} m, **kpc**, **Mpc** (see later discussion)
mass	kg	gm	$M_\oplus$ = 6.0x10^{24} kg $M_\odot$ = 2.0x10^{30} kg
current	A = C/s	e.s.u. /s (=1)	
angle	rad	rad	**deg, arcmin, arcsec**
solid angle	sr	sr	**deg^2, arcmin^2, arcsec^2**
magnetic field	T	G (=10^{-4}T)	[usually use **G** or **μG**]
velocity	m/s	cm/s	**km/s** (see later discussion)

- **Astronomical Unit** (AU): Earth's orbit radius.
- **Parsec:** (pc): The distance at which 1 AU subtends an angle of one arcsecond due to parallax.
- **Parallax:** Using the baseline of earth's orbit to detect total angular shift of nearby stars relative to much more distant stars.
$$d = \frac{R_\oplus}{\tan\frac{\theta}{2}} = \frac{R_\oplus}{\tan p}.$$
- Other methods of measuring distances are: Measuring time delay between close and far part of super nova ring, Observing time lag between two gravitationally lensed images, Studying its radiation (something depends on length $\times$ density^2, others on length $\times$ density).
- Relative distances of stars who's (relative) luminosities L and (relative) brightnesses f are known are calculated as follows:
$$\frac{d_1}{d_2} = \sqrt{\frac{f_2}{f_1}\frac{L_1}{L_2}}$$
- **Doppler effect:** The radial velocity is easy to measure using the Doppler shift of light:
$$\frac{v_{\text{em}}}{v_{\text{obs}}} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}, \quad \text{Non-relativistic } v \ll c\text{: } \frac{\Delta v}{v} \sim \frac{\Delta\lambda}{\lambda} \sim \frac{v}{c}$$
  · Moving towards us, frequency increases →*blueshift*.
  · Moving away from us, frequency decreases →*redshift*.
  Radial velocity ($\dot{r}$) can be measured very accurately, but angular velocities ($\dot{\theta}, \dot{\varphi}$) are very hard to measure. This often effectively reduces the 6-dimensional phase-space to only 3 measurable components: $(r, \theta, \varphi, \dot{r}, \dot{\theta}, \dot{\varphi}) \to (\theta, \varphi, \dot{r})$.
- **Spherical coordinates:**



  For rotating systems sometimes the equator is defined to be $\theta = 0°$ and the poles $\pm 90°$ instead of the equator at $\theta = 90°$ and the poles at $\theta = 0°$ and $\theta = 180°$.
  · Angle: $\mathrm{d}s = r\,\mathrm{d}\gamma = r(\mathrm{d}\theta + \sin\theta\,\mathrm{d}\varphi)$.
  · Solid angle: $\mathrm{d}A = r^2\,\mathrm{d}\Omega = r^2(\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi)$.
  If θ defined up/down the equator:
  · Angle: $\mathrm{d}s = r\,\mathrm{d}\gamma = r(\mathrm{d}\theta + \cos\theta\,\mathrm{d}\varphi)$.
  · Solid angle: $\mathrm{d}A = r^2\,\mathrm{d}\Omega = r^2(\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi)$.
  Integrating over all γ or all Ω we get $\int \mathrm{d}\gamma = 2\pi$, $\int \mathrm{d}\Omega = 4\pi$.
  In general:
$$\mathrm{d}\Omega = \frac{\vec{r}\cdot\vec{n}\,\mathrm{d}A}{r^2}$$

2 Radiation and Matter

At any/every point in space we have an electromagnetic radiation field that describes the number and frequency of the photons passing through that location in different directions.

2.1 Description of radiation fields

- **Isotropic radiation field:** The number of photons coming from different directions is always the same. If the number of photons varies depending on direction it is **anisotropic**.
- The **cosmic microwave background** (CMB) is an isotropic 2.736 K black body radiation field. It's energy density U_ν (the energy contained per unit volume) within a small interval of frequency $\mathrm{d}\nu$ given by *Planck's Law*:
$$U_\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$
- The **Extragalactic background light** (EBL) is the sum of all the light of all galaxies in a given direction. It is also highly isotropic.
- **Specific Intensity** I_ν**:** Defined in terms of the energy $\mathrm{d}E_\nu\,\mathrm{d}\nu$ that passes through an area $\mathrm{d}A$ in time $\mathrm{d}t$ within $\mathrm{d}\nu$ within $\mathrm{d}\Omega$ in direction $\vec{n}$ oriented at angle θ to the normal of $\mathrm{d}A$:
$$\mathrm{d}E_\nu\,\mathrm{d}\nu = I_\nu(\vec{n}, \nu)\cos\theta\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu.$$
- **Radiation Flux** F_ν**:** I_ν integrated over all directions:
$$\boxed{F_\nu = \int I_\nu \cos\theta\,\mathrm{d}\Omega}.$$
  In the isotropic case the net energy flux must be zero.
- **Total energy flux** F**:** F_ν integrated over all frequencies:
$$F = \int F_\nu\,\mathrm{d}\nu.$$
- **Energy density** $\mathrm{d}U_\nu$**:** Energy $\mathrm{d}E_\nu$ that passes through $\mathrm{d}A$ during $\mathrm{d}t$:
$$\mathrm{d}U_\nu = \frac{\mathrm{d}E_\nu}{\cos\theta c\,\mathrm{d}t\,\mathrm{d}A} = \frac{I_\nu(\vec{n})}{c}\,\mathrm{d}\Omega.$$
  Integrated over all directions:
$$\boxed{U_\nu = \int \frac{I_\nu(\vec{n})}{c}\,\mathrm{d}\Omega} \quad \text{isotropic:} \quad U_\nu = \frac{4\pi}{c}I_\nu.$$
- Specific intensity I_ν of an **Black-body radiation** field:
$$I_{\nu,\text{black-body}} =: B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$
- The **Radiation pressure** is given by the flux of momentum perpendicular to the surface:
$$\mathrm{d}P_\nu = \frac{\mathrm{d}E_\nu\cos\theta}{c\,\mathrm{d}A\,\mathrm{d}t} = \frac{I_\nu}{c}\cos^2\theta\,\mathrm{d}\Omega \qquad \boxed{P_\nu = \frac{1}{c}\int I_\nu\cos^2\theta\,\mathrm{d}\Omega}$$
  And for isotropic fields:
$$P_\nu = \frac{4\pi}{3}\frac{I_\nu}{c}.$$
  We can also relate the pressure and energy density of any isotropic radiation field:
$$P_\nu = \frac{1}{3}U_\nu \qquad P = \frac{1}{3}U.$$
- In a vacuum, the specific intensity along a light path does not change (in a non-expanding universe):
$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0.$$

2.2 Observing radiation from distant sources

When observing a distant point source astronomers will orient the telescope aperture of area A perpendicularly to the direction $\vec{n}$, so $\theta = 0$ and $\cos\theta = 1$.

- The **Flux density** f_ν is energy per area, per time interval, per frequency ($\mathrm{Wm^{-2}Hz^{-1}}$) and satisfies:
$$E_\nu\,\mathrm{d}\nu = f_\nu A\,\mathrm{d}\nu\,\mathrm{d}t.$$
- The **spectral resolution** R of an instrument is given by:
$$R = \frac{\lambda}{\Delta\lambda_{\text{inst.}}} = \frac{\nu}{\Delta\nu_{\text{inst.}}}.$$
- The average over some interval $\Delta\lambda$, $\Delta\nu$ produces a **flux** f:
$$f = \int_{\nu_1}^{\nu_2} f_\nu\,\mathrm{d}\nu = \int_{\lambda_1}^{\lambda_2} f_\lambda\,\mathrm{d}\lambda$$
$$\nu f_\nu = \lambda f_\lambda \qquad \nu\lambda = c \;\to\; \mathrm{d}\nu = (-)\frac{c}{\lambda^2}\,\mathrm{d}\lambda.$$
- The **Luminosity** L is the total radiation emitted by a source. For isotropically radiating sources L it is given by:
$$L_{\nu,\lambda} = 4\pi d^2 f_{\nu,\lambda}\ [\mathrm{W\,Hz^{-1}, W\,Å^{-1}}] \qquad L = 4\pi d^2 f\ [\mathrm{W}].$$
  $L_\odot = 3.83\times 10^{26}$ W is a shorthand for the solar luminosity.
- **Magnitudes** m measure the flux density f_ν within a number of more or less standard filter passbands i. For historic reasons magnitudes are measured logarithmically relative to the bright start Vega:
$$m_i = -2.5\log_{10}\frac{f_{\nu,i}}{f_{\nu,i,\text{Vega}}}.$$
- **Absolute Magnitudes** M refer to the (apparent) magnitude of a source if it was placed 10 pc away:
$$m_i = M_i + 5\log_{10}\left(\frac{d}{1\,\text{pc}}\right).$$

2.3 Some important radiation mechanisms

- For **thermal emission** we require an object that is considered to be in thermodynamic equilibrium. The emergent flux at the surface is given by the Stefan-Boltzmann law:
$$F = \sigma T_{\text{eff}}^4, \quad \text{where } \sigma = \frac{2\pi^5 k_{\text{B}}^4}{15h^3c^2} = 5.67\times10^{-8}\,\mathrm{Jm^{-2}s^{-1}K^{-4}}.$$
  The thermal radiation will have black body for, where the peak of the Planck function is given by Wien's displacement law:
$$\lambda_{\text{peak}} = \frac{b}{T}, \quad \text{where } b = 2898\,\mu\text{m K}.$$
  The energy density of the radiation U is given by
$$U = \frac{4}{c}\sigma T^4 = a_B T^4$$
- **Atomic line emission** occurs when atoms (primarily their electrons) change energy states by absorbing a photon, colliding, ionization or other. Atomic energy transitions are typically around $\Delta E \approx$ few eV. The average kinetic energy in a gas of Temperature T is given by Boltzmann's constant
$$\langle E\rangle = \frac{3}{2}k_{\text{B}}T.$$
  For collisions to reach eV levels we require temperatures of $T \approx 10^4$ K.

- The rate of spontaneous de-excitation by photon emission is given by the **Einstein** A $[\mathrm{s}^{-1}]$. *Permitted* transitions have large A, *forbidden* transitions have small A.

- Emission lines are denoted as the ionic species with the ionization state in roman numerals +1 followed by the wavelength, e.g. [OII] 3726, 3729.

- Cool gases ($T \ll 10^4\,\mathrm{K}$) mostly cannot excite electrons out of their ground state, but the molecules still have several rotational and vibrational quantum energy states at the meV level, that can be excited. This gives rise to **Molecular line emission**.

- Rotational transitions are equally spaced in frequency:

$$E_{\mathrm{rot}} = \frac{J(J+1)}{2I}\hbar^2, \qquad \Delta E_{J,J-1} = \frac{J}{I}\hbar^2.$$

  Their Einstein A is usually very small $A^{-1} \approx 10^7\,\mathrm{s}$, a few months. The J levels in the gas reflect the thermodynamic equilibrium.

- The **21-cm line emission** comes from the slight energy difference of a Hydrogen atom with the protons and electrons spin being parallel versus anti-parallel. Its Einstein $A \approx 2.85 \times 10^{-15}\,\mathrm{s}^{-1}$ is very small, so the radiative half-life is about 11 million years.

- One form of continuum emission is **synchrotron emission**, which comes from the radiation of a relativistically accelerated electron in a magnetic field. The radiated power of an accelerated charge is given by Larmor's equation:

$$P = \frac{2}{3}\frac{q^2\dot{v}^2}{4\pi\varepsilon_0 c^3}$$

  The gyro frequency ω_{G} of an electron spiralling around a magnetic field line is given by

$$\omega_{\mathrm{G}} = \frac{qB}{m} \approx 1.76 \times 10^6\,\mathrm{rad\,s^{-1}} \cdot B(G),$$

  where $G = 10^{-4}\,\mathrm{T}$ is a Gauss. Typical interstellar magnetic fields have only $\mu\mathrm{G} \approx 10^{-10}\,\mathrm{T}$ strength. Therefore synchrotron emission is most prominent at radio frequencies around $100\,\mathrm{MHz} - 10\,\mathrm{GHz}$.

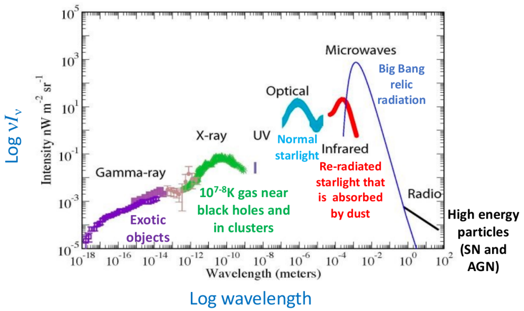
- In very hot plasma ($T \approx 10^8\,\mathrm{K}$) most radiation comes from **thermal bremsstrahlung**, where fast moving electrons get redirected when passing close to a nucleus and radiate. The total emissivity (radiated power per volume per $\mathrm{d}\nu$) for a plasma of temperature T, free electron density n_e and density of ions n_i of charge Z_e is given by

$$\varepsilon_\nu = 6.8 \cdot 10^{-51}\frac{n_e n_i Z^2}{T^{1/2}} e^{\frac{-h\nu}{kT}} g(\nu, T),$$

  where $g(\nu, T)$ is a Gaunt factor. Bremsstrahlung emission extends up to X-ray frequencies ($\lambda \sim \mathrm{nm}$, $E \sim \mathrm{keV}$).

2.4 The universe at different wavelengths

- The **Extragalactic Background Light** (EBL) is a measure of the overall radiation field of the universe. If we go to a random extragalactic point in space, we see a more or less isotropic radiation field. By measuring the specific intensity I_ν and averaging over all directions we get the EBL.



2.5 Non-electromagnetic information

- We can also gain information about the universe by detecting neutrinos, cosmic rays and gravitational waves.

3 Radiative Transfer and Stellar Atmospheres

3.1 Radiative transfer

- The basic **radiative transfer equation** describes the interaction of matter and radiation:

$$\boxed{\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu},$$

  where j_ν $\left[\mathrm{time}^{-1}\mathrm{length}^{-3}\mathrm{time}\right]$ and α_ν $\left[\mathrm{length}^{-1}\right]$ are emission and absorption coefficients. α_ν is given by the total cross-section for absorption σ of each particle per unit volume: $\alpha_\nu = \sum_i n_i\sigma_i$.

- The **optical depth** τ_ν is given by

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s \qquad \tau_\nu(s) = \int_{s_0}^{s}\alpha_\nu(s')\,\mathrm{d}s'.$$

  In case of no emission ($j_\nu = 0$) the intensity of radiation decreases exponentially:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}.$$

  · $\tau_\nu \ll 1$: optically thin, transparent
  · $\tau_\nu \gg 1$: optically thick, opaque

- The **source function** S_ν is the ratio of the emission to absorption coefficients:

$$S_\nu = \frac{j_\nu}{\alpha_\nu}.$$

  Using it we can rewrite the radiative transfer equation:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu.$$

  · $S_\nu - I_\nu > 0$: I_ν increases
  · $S_\nu - I_\nu < 0$: I_ν decreases
  $\longrightarrow$ I_ν tends towards S_ν

- The general solution for $I_\nu(\tau_\nu)$ is:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}S_\nu(\tau_\nu')\,\mathrm{d}\tau_\nu'.$$

  If the physical properties S_ν are constant this reduces to:

$$I_\nu(\tau_\nu) = \underbrace{I_\nu(0)e^{-\tau_\nu}}_{\text{absorption}} + \underbrace{S_\nu(1 - e^{-\tau_\nu})}_{\text{emission}}.$$

  If we are only interested in the emergent radiation we can make the following approximations:
  · optically thin $\tau_\nu \ll 1$: $\quad I_\nu(\tau_\nu) = S_\nu\tau_\nu \sim j_\nu s$
  · optically thick $\tau_\nu \gg 1$: $\quad I_\nu(\tau_\nu) = S_\nu$

- **Kirchhoff's law** states that if matter is in thermodynamic equilibrium, then the source function $S_\nu(\nu)$ is equal to the black-body radiation:

$$S_\nu = B_\nu(T) \qquad \Leftrightarrow \qquad j_\nu = \alpha_\nu B_\nu(T).$$

  From this follows that for optically *thick* sources in thermodynamic equilibrium, the emergent I_ν will be given by the Planck function B_ν, and not show any peaks as is the case with optically *thin* sources.

3.2 Thermodynamic equilibrium

- The velocities of the particles in a gas in thermodynamic equilibrium follow the **Maxwell distribution**:

$$\mathrm{d}n_v\,\mathrm{d}v = 4\pi n\left(\frac{m}{2\pi k_B T}\right)v^2\exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v.$$

- In thermodynamic equilibrium the number of atoms in an excited state with energy E is given by the **Boltzmann distribution Law**:

$$\frac{n_E}{n_0} = \exp\left(-\frac{E}{k_B T}\right).$$

- The **Saha formula** gives the fraction of atoms that are ionized x at a given temperature T, pressure P and for the required ionization energy χ:

$$\frac{x^2}{1-x} = \frac{(2\pi m_e)^{\frac{3}{2}}}{h^3}\frac{(k_B T)^{\frac{5}{2}}}{P}\exp\left(-\frac{\chi}{k_B T}\right).$$

- Something is considered to be in thermodynamic equilibrium, if the temperature does not change along the distance of a *mean free path*. This is the case, both for atom and photons. For photons the mean free path is exactly α_ν^{-1}. If the temperature does not vary along both the collisional and electromagnetic mean free path, we have **Local Thermodynamic Equilibrium**. Note though, that both may not be fulfilled.

3.3 Radiative transfer in stellar atmospheres

- The **plane parallel atmosphere** is a small region far away from the center of the star with a Cartesian coordinate system, where all properties are constant on the xy-plane. θ is the angle to the z-axis, $\mu = \cos\theta$ and therefore we re-write the radiative transfer equation:

$$\mu\frac{\partial I_\nu(z,\mu)}{\partial z} = J_\nu - \alpha_\nu I_\nu, \qquad \mu\frac{\partial I_\nu(z,\mu)}{\partial \tau_\nu} = I_\nu - S_\nu.$$

  We also set $\mathrm{d}\tau_\nu = -\alpha_\nu\,\mathrm{d}z$, so that τ reflects the vertical depth.

- I_ν at different depths (τ) and different directions (μ) is given by:

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu}.$$

  The I_ν measured at some point reflects the conditions that occur at a distance $\Delta\tau = 1$ from that point.

- If some quantity A is a function of $\mu = \cos\theta$ only, then the integral over Ω is given by $\int A(\mu)\,\mathrm{d}\Omega = 2\pi\int_{-1}^{1}A(\mu)\,\mathrm{d}\mu$:

$$U_\nu = \frac{2\pi}{c}\int_{-1}^{1}I_\nu\,\mathrm{d}\mu \quad = \frac{4\pi}{c}B_\nu(\tau_\nu)$$
$$F_\nu = 2\pi\int_{-1}^{1}I_\nu\mu\,\mathrm{d}\mu \quad = \frac{4\pi}{3}\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu}$$
$$P_\nu = \frac{2\pi}{c}\int_{-1}^{1}I_\nu\mu^2\,\mathrm{d}\mu = \frac{4\pi}{3c}B_\nu(\tau_\nu)$$

- Using the Stefan-Boltzmann law $F = \sigma T_{\mathrm{eff}}^4$ we get the ratio between the anisotropic and isotropic terms:

$$\frac{\text{anisotropic}}{\text{isotropic}} = \frac{3}{4}\left(\frac{T_{\mathrm{eff}}}{T}\right)^4$$

- In a **grey atmosphere** we assume that α_ν is independent of ν. We then get the following:

$$\mathrm{d}\tau = -\alpha\,\mathrm{d}z \qquad \mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I - S$$
$$J := \frac{1}{2}\int_{-1}^{1}I\,\mathrm{d}\mu = \frac{c}{4\pi}U \qquad \frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{F}{c} \qquad J = S$$

  With this model we can also rather accurately predict the effect of **limb darkening**.

- The emergent radiation from the surface of a star has the characteristics of black-body radiation. It has the characteristics of the layer that lies $\Delta\tau$ below the surface, the so called **photosphere** (for our sun around 5800 K). Above the surface lies the **corona** consisting of very hot ($<10^6$K) but very low density (10^{-19}Kg m^{-3}) material. This region is very optically thin $\tau \ll 1$ and therefore has absorption lines.

3.4 Radiative energy transport in the interior of stars

- In the stellar interior energy is being injected from fusion reactions. The total radiation flux outwards in terms of the gradient of the radiation pressure is given by:

$$F = -\frac{c}{\alpha_R}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{a_B}{3}T^4\right), \quad \text{where } \frac{1}{\alpha_R} = \frac{\int\frac{1}{\alpha_\nu}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}.$$

3.5 Opacity

- The **opacity** χ of the material is defined in terms of the mass density ρ. It gives the average absorption coefficient per unit mass density:

$$\alpha_R = \rho\chi.$$

- Opacity is low at low temperatures, then rises to a peak and then declines again to a more or less constant value. Kramer's' Law provides a rough approximation for the falling part of the curve:

$$\chi \propto \frac{\rho}{T^{3.5}}.$$

- The opacity decreases for low temperatures, because the electrons become ionized, which then interact strongly with the electromagnetic radiation. This is called **Thompson scattering**. The opacity due to it is given by

$$\chi_T = \frac{n_e}{\rho}\sigma_T, \quad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2 \approx 6.65\times10^{-29}\,\mathrm{m}^2.$$

- If the electron is tightly bound to the atom, we get **Rayleigh scattering**.

4 Stellar Structure

4.1 Equations of stellar structure

- M_r is the mass enclosed within a radius r, so

$$\boxed{\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho}$$

- For **hydrostatic support** we require the gravitational force to be balanced by the pressure gradient:

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho}$$

- Rough approximations yield the central pressure P_0 and central temperature T_0:

$$P_0 \approx \frac{3}{2\pi}\frac{GM^2}{R^4} \approx 6\times10^{14}\,\mathrm{N/m^2}$$
$$T_0 \approx \frac{1}{2}\frac{GMm_H}{k_B R} \approx 10^7\,\mathrm{K}$$

- The **Virial theorem** relates the thermal energy E_T to the gravitational energy E_G of any self gravitating system. For gases with the thermal energy density given by $\frac{3}{2}k_B T = \frac{3}{2}P$ we have

$$2E_T = -E_G, \qquad E_{\text{tot}} = \frac{1}{2}E_G = -E_T.$$
$$\mathrm{d}E_{\text{tot}} = -\,\mathrm{d}E_T \quad \text{(negative heat capacity)}$$

For relativistic particles the energy density is $3P$, therefore

$$E_T = -E_G, \qquad E_{\text{tot}} = 0.$$

- L_r is the **total energy flux** integrated over $4\pi r^2$, which flows outward within a star at radius r. ε is the energy production rate per unit mass, per unit time. The change in L_r in r is therefore

$$\boxed{\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2\rho\varepsilon}$$

- The energy transport by **radiation** $L_r = 4\pi r^2 F$ is given by:

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4a_B c}\frac{\rho\chi}{T^3}\frac{L_r}{4\pi r^2}}$$

High $\rho\chi = \alpha_R \to$ short mean free path $\to$ requires high temperature gradient to drive outward flow of L_r.

- For energy transport by **convection** we look at adiabatic changes in r of small packets of gas of pressure P and density ρ. Depending on how the new P^* and ρ^* relate to the surrounding P' and ρ' (buoyancy), the packet will return to its position (stable) or move further away (unstable). For a gas to be stable against convective currents, it has to fulfill the **Schwarzschild stability criterion**:

$$\boxed{\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right| \lesssim \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{\mathrm{d}P}{\mathrm{d}r}\right|}$$

Convection currents will ensure that the conditions adapt as to just maintain that condition.
High $\rho\chi = \alpha_R \to$ large temperature gradient $\to$ more convection.

	core	atmosphere
Low mass	radiative	convective
High mass	convective	radiative

4.2 Stellar structure models

- The chemical composition is described by the mass fraction X_i of the various elements i. Hydrogen is written X, Helium Y and all heavier elements ("Metals") as Z ("metallicity"). Note $X+Y+Z=1$.
- We want equations for the pressure $P(\rho,T,X_i)$, the opacity $\chi(\rho,T,X_i)$ and the energy production rate $\varepsilon(\rho,T,X_i)$.
- For the equation of state $P(\rho,T,X_i)$ we assume the stars to be perfect gases. The number density of particles in a fully ionized gas is given by the number nuclei and electrons:

$$n = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)\frac{\rho}{m_H}.$$

$$P = nkT = \frac{k_B}{\mu m_H}\rho T, \qquad \text{where } \mu = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}.$$

- The energy production rate $\varepsilon(\rho,T,X_i)$ we get from understanding fusion, and the opacity $\chi(\rho,T,X_i)$ is very complex. Given those we can then solve for a unique solution with the four boundary conditions:

$$\text{For: } r=0: \qquad M_r = 0, \quad L_r = 0$$
$$\text{For: } r=R: \qquad \rho = 0, \quad T = 0$$

- With gross simplifications we can derive the following relations:

$$T \propto \frac{M}{R}, \quad L \propto R^2 T_{\text{eff}}^4, \quad L \propto M^3, \quad M \propto T_{\text{eff}}^2, \quad L \propto T_{\text{eff}}^6.$$

- Chemical reactions (combustion of substances with oxygen) we first though to be the sources of stellar energy ($\sim$ eV). But with a maximum of 10^{38} J in our sun, which has a luminosity of around 3.8×10^{26} W, they could only power it for around 10^4 years.
- Secondly it was considered that the sun was powered by continually reducing its radius R and therefore increasing its (negative) gravitational potential energy. This would power the sun for around 10^7 years (**Kevin-Helmholtz timescale**) for each halfing of its size.
- The real sources of the suns energy are nuclear reactions ($\sim$ MeV). As the luminosity ultimately reflects the rate of consumption of the nuclear fuel, we get the following scaling between the mass and the lifetime τ:

$$\tau \propto M^{-2}.$$

- We looked at two sources of pressure: Thermal pressure (non-relativistic) and radiation pressure:

$$P_{\text{gas}} = \frac{\rho}{\mu m_H}k_B T, \qquad P_{\text{rad}} = \frac{1}{3}a_B T^4.$$

Introducing the thermal pressure fraction β, we get

$$P_{\text{gas}} = \beta P, \qquad P_{\text{rad}} = (1-\beta)P$$

$$P = \left(\frac{3(1-\beta)}{\beta^4 a_B}\right)^{\frac{1}{3}}\left(\frac{k_B\rho}{\mu m_H}\right)^{\frac{4}{3}}.$$

For low mass stars $\beta\approx1$, for about $20\,\mathrm{M}_\odot$ we have $\beta\approx0.85$, and for $100\,\mathrm{M}_\odot$ we have $\beta\approx0.5$.

4.3 Nuclear fusion

- Atomic nuclei are very tightly bound because of the very attractive, short range **Strong Force** (ineffective beyond 10^{-15} m). They have smaller mass than the sum of the masses of their constituent protons and neutrons:

$$m_{\text{nuc}} < (Zm_p + (A-Z)m_n).$$

The mass deficit is the binding energy:

$$E_B = (Zm_p + (A-Z)m_n - m_{\text{nuc}})c^2.$$

- The **binding energy per nucleon**

$$f = \frac{E_B}{A},$$

is shown in the figure below. The binding energy starts to decrease after ^{56}Fe, because the nucleus gets too large.

- For two particles to get close enough to feel the strong force they have to cross the potential barrier given by the electrostatic coulomb potential energy

$$E_C = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{r}.$$

At typical kinetic energies of particles with Z_1 and Z_2 protons at $T\sim10^7$ K are only of order keV, whereas the barrier is around Z_1Z_2MeV. They can still pass through the barrier though, via quantum tunneling.

- If we assume the relative velocities of the particles to follow a Maxwellian distribution, and take into account quantum tunneling, we get an overall reaction rate r that is strongly peaked around the **Gamow peak** E_0:

$$E_0 = \left[\left(\frac{m}{2}\right)^{1/2}\frac{Z_1 Z_2 e^2 k_B T}{4\varepsilon_0\hbar}\right]^{2/3}.$$

- The new composite nucleus will be excited by $\Delta\mathcal{E}$ above the ground state. The energy production rate ε (per unit mass) is then given by

$$\rho\varepsilon = r\Delta\mathcal{E}.$$

- One way to fuse four H into ^{4}He is the **p-p chain**:

$$^1\mathrm{H} + {}^1\mathrm{H} \longrightarrow {}^2\mathrm{H} + \mathrm{e}^+ + \nu$$
$$^2\mathrm{H} + {}^1\mathrm{H} \longrightarrow {}^3\mathrm{He} + \gamma$$

Then it will go through either the *pp1* (most likely), *pp2* or *pp3* process:

$$\textit{pp1:}\quad {}^3\mathrm{He} + {}^3\mathrm{He} \longrightarrow {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H}$$

$$\textit{pp2:}\quad {}^3\mathrm{He} + {}^4\mathrm{He} \longrightarrow {}^7\mathrm{Be} + \gamma$$
$${}^7\mathrm{Be} + \mathrm{e}^- \longrightarrow {}^7\mathrm{Li} + \nu$$
$${}^8\mathrm{Li} + {}^1\mathrm{H} \longrightarrow {}^4\mathrm{He} + {}^4\mathrm{He}$$

$$\textit{pp3:}\quad {}^3\mathrm{He} + {}^4\mathrm{He} \longrightarrow {}^7\mathrm{Be} + \gamma$$
$${}^7\mathrm{Be} + {}^1\mathrm{H} \longrightarrow {}^8\mathrm{B} + \gamma$$
$${}^8\mathrm{B} \longrightarrow {}^8\mathrm{Be} + \mathrm{e}^+ + \nu$$
$${}^8\mathrm{Be} \longrightarrow {}^4\mathrm{He} + {}^4\mathrm{He}$$

The net effect for all of them is the same:

$$4\,{}^1\mathrm{H} \longrightarrow {}^4\mathrm{He} + 2\,\mathrm{e}^+ + 2\,\nu^+\text{energy},$$

with a very strong temperature dependence on the energy production rate:

$$\varepsilon_{pp} = 0.24\rho X^2 T_6^{-2/3}\exp\left(-33.8T_6^{-1/3}\right)\mathrm{W\,kg^{-1}}.$$

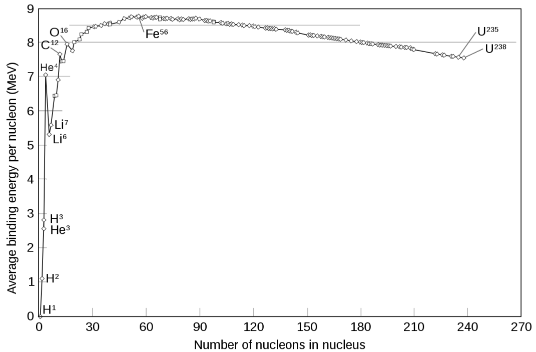
- If carbon nuclei are present, the **CNO cycle** is also possible:

$$^{12}\mathrm{C} + 4\,{}^1\mathrm{H} \longrightarrow \cdots \longrightarrow {}^{12}\mathrm{C} + {}^4\mathrm{He} + \text{energy}.$$

It has an even stronger temperature dependence:

$$\varepsilon_{\text{CNO}} = 8.7\times10^{20}\rho X_{\text{CNO}} X T_6^{-2/3}\exp\left(-152.3T_6^{-1/3}\right)\mathrm{W\,kg^{-1}}.$$

- These reactions release a lot of neutrinos ν, which can be detected here on earth.



5 Stellar Evolution

5.1 Nucleosynthesis beyond ^{4}He

- There are no nuclei between ^{4}He and ^{12}C that are more tightly bound than ^{4}He. Further, there are no stable nuclei with atomic mass numbers $A = 5$ or 8. Therefore, fusion of two ^{4}He or adding a single proton to ^{4}He will not work.
- Some ^{4}He will collide and endothermically (91.8 keV) form a very short lived (3×10^{-16} s) ^{8}Be nucleus. If the production of ^{8}Be is high enough and an equilibrium is reached, a ^{8}Be

might be hit by a ^{4}He nucleus to create an excited ^{12}C*, which is 350 keV above the binding energy of the original three ^{4}He.

$$^4\text{He} + {}^4\text{He} \longrightarrow {}^8\text{Be}$$
$$^8\text{Be} + {}^4\text{He} \longrightarrow {}^{12}\text{C}^* \xrightarrow{-7.65\,\text{MeV}} {}^{12}\text{C}$$

- The new ^{12}C can then exothermically react with ^{4}He:

$$^{12}\text{C} + {}^4\text{He} \longrightarrow {}^{16}\text{O} + \gamma$$

Fortunately there is no excited quantum state of ^{16}O* nearby, therefore the destruction of ^{12}C is hindered. This is where the relative abundance of carbon and oxygen (1:3) stems from.

- Further fusion reactions up to Iron require higher and higher temperatures. Above 5×10^8 K we get carbon-burning, above 10^9 K neon-burning and above 2×10^9 K we bet oxygen-burning. A lot of the later reactions involve ^{4}He, which is why we have more elements that are composed of multiples of ^{4}He.

5.2 Degeneracy Pressure

- As the temperature decreases, the momentum of the particles goes towards zero. But no two fermions can occupy exactly the same quantum state (**Pauli exclusion principle**), therefore they are forced into higher momentum state. This gives rise to **degeneracy pressure**.

- Lower mass particles have smaller momentum and will have higher density in momentum space. Electrons are therefore affected by the Pauli exclusion principle way before protons or neutrons.

- To minimize the total energy we must occupy all momentum states starting from the minimum up to some limit, the **Fermi momentum** p_F:

$$p_\text{F} = \left(\frac{3h^3 n_e}{8\pi}\right)^{\frac{1}{3}}, \qquad n_e = \frac{\rho}{\mu_e m_H},$$

where ρ is the (mass-)density and μ_e is the mean atomic mass of the nucleons associated with each electron.

- The pressure is then given by

$$P = \frac{1}{3}\int v p f(p) 4\pi p^2\,\mathrm{d}p = \frac{8\pi}{3}\int_0^{p_\text{F}} \frac{p^4c^2}{\sqrt{p^2c^2+m^2c^4}}\,\mathrm{d}p,$$

where the distribution function of the momentum $f(p)$ (usually given by the Maxwellian distribution) is here given by $f(p) = \frac{2}{h^3}$. For *non-relativistic* matter this yields

$$P = \frac{8\pi}{15h^3m_e}p_\text{F}^5 = \frac{3^{2/3}}{20\pi^{2/3}}\frac{h^2}{m_e}n_e^{5/3},$$

and for *relativistic* particles

$$P = \frac{2\pi c}{3h^3}p_\text{F}^4 = \frac{3^{1/3}}{8\pi^{1/3}}hcn_e^{4/3}.$$

- In an object with number density n, the maximum available volume for a particle is n^{-1}. The **Heisenberg uncertainty principle** $\Delta x\Delta p \sim \hbar$ then also limits the minimum momentum to $p \sim \hbar n^{1/3}$.

- The hydrostatic support equation can take the very simple form of the **Lane-Emden equation**, where θ and ξ are dimensionless variables describing the density and the radius:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n.$$

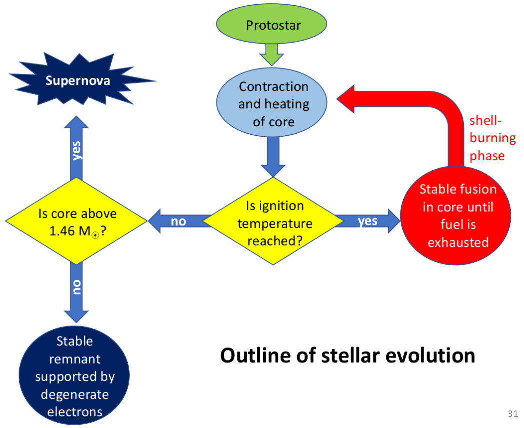
The solution to this equation gives the following relations:

non-relativistic:	$R \propto \rho_c^{-1/6}$,	$M \propto \rho_c^{1/2}$,	$R \propto M^{-1/3}$
relativistic:	$R \propto \rho_c^{-1/3}$,	$M \propto \rho_c^{0}$,	$R \propto M^{-\infty}$

- We can see that as the mass increases, the particles are forced into higher and higher energy states. The limits the maximum mass that can be supported, beyond this maximum the object will collapse. This mass is given by the **Chandrasekhar mass** $M_\text{Ch} = 1.46\left(\frac{2}{\mu_e}\right)^2 \text{M}_\odot \approx 1.46\text{M}_\odot$.

- Degeneracy pressure is a source of pressure that is completely independent of temperature and therefore immune to the continuous loss of energy. If of constant mass, an object will remain at constant size, slowly radiating away its thermal energy. Degeneracy therefore "stops" the evolution of a star.

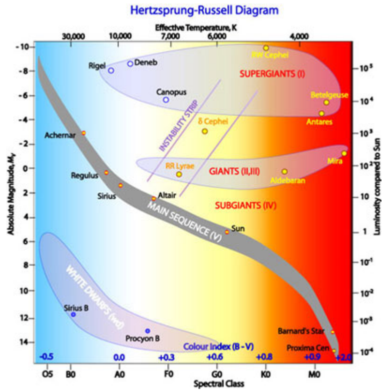
5.3 The Hertzsprung-Russel diagram and phases of stellar evolution



- The **Hertzsprung-Russel diagram** plots the luminosity (absolute magnitude) against the temperature (color). As a star evolves it moves around in the diagram and leaves an evolutionary track.

- **Brown dwarfs** are proto-stellar objects with masses below $0.08\,\text{M}_\odot$. They are compact, cool, dim objects which have become supported by degeneracy pressure before they ever got hot enough to start fusing H. They sometimes burn small amounts of deuterium ^{2}H.

- **Red giants** are large, luminous, cool stars who's fuel is largely exhausted in the core. It therefore slowly shrinks and heats up while fusion continues in the surrounding shell. This increases the total luminosity and the radius of the star, while decreasing the surface temperature. It therefore ascends to the **red giant branch**. After it has heated up and started fusing Helium again, it moves over onto the **horizontal branch**.



- The **instability strip** on the HR diagram represents starts, who's opacity in the atmosphere oscillates. A change in the opacity drives a change in the temperature-density, which in turn affects the opacity and so on. The most famous examples are Cepheid stars.

- **White dwarfs** are compact objects support by degenerate electron pressure with inert cores that have ceased fusion, because the didn't get hot enough. They only cool very slowly and will remain hot very long, illuminating potential dust around them creating **planetary nebulae**. It is though, that all stars with masses less than $8\,\text{M}_\odot$ will become white dwarfs.

- **Very massive stars** are stars with masses above $8\,\text{M}_\text{o}$dot. Their cores will become so dense that the electrons already become degenerate while it is still fusing. Once the fuel runs out though, degeneracy pressure is not enough to support the star (as it has mass greater that the Chandrasekhar limit of $1.4\,\text{M}_\odot$), and it will collapse.

5.4 Supernova explosions

- Free neutrons are usually unstable and decay into a proton an electron with the emission of an anti-neutrino and a half-life of about 10.2 minutes. However, when electrons become degenerate and are forced into higher energy states, it will be energetically favorable for an electron to combine with a proton to make a neutron.

$$n \longrightarrow p^+ + e^- + \bar{\nu}_e, \qquad e^- + p^+ \longrightarrow n + \nu_e$$

- This happens once the Fermi energy E_F (associated to the Fermi momentum p_F) exceeds the excess mass of the neutron. This gives us a critical Fermi momentum $p_{\text{F},c}$ and critical density of electrons $\rho \approx 1.2\times10^{10}\,\text{kg/m}^3$:

$$E_\text{F} = \sqrt{p_\text{F}^2c^2 + m_e^2c^4} > (m_n - m_p - m_e)c^2$$

$$\rightarrow \quad p_{\text{F},c} = m_ec\left[\left(\frac{m_n-m_p}{m_e}\right)^2 - 1\right]^{1/2}$$

- As neutrons are way more massive than electrons, we need extremely high densities if we want to create pressure from degenerate neutrons. Such an object is called a **neutron star**. As now the strong force and general relativity are relevant, calculations become very complicated.

- A neutron star of mass around $1.5\,\text{M}_\odot$ has an extent of 10 km, only a few times of the Schwarzschild radius.

- The Chandrasekhar mass for neutron degeneracy is not well known, but thought to be around $2-3\text{M}_\odot$.

- When the core of a star collapses down to a size of 10 km it releases potential energy on the order of 10^{46} J within a few seconds. Most of this energy is in the form of energetic neutrinos, 1 % of which (10^{44} J) is reabsorbed in the outer envelope of the star and converted back into kinetic energy, creating an explosion of material outwards.

- The Milky Way should have around one core collapse every 100 years, and there should be around one core collapse every second in the observable universe.

- Nucleosynthesis beyond ^{56}Fe is endothermic and is therefore not done through fusion, but rather through the capture of free neutrons, which being electrically neutral, do not face the Coulomb barrier. Free neutrons are only around during the advanced burning stages and during the collapse to a neutron star.

- Neutrons get absorbed into nucleons via either the α-decay ($\Delta A = -4$, $\Delta Z = -2$), called the *p*-process, or the β-decay ($\Delta A = 0$, $\Delta Z = +1$), called the *s*-process and keeps the nuclei closer to the stability strip in the *A*-*Z* plane.

- All elements above Bismuth (including Uranium $Z = 92$ and Gold $Z = 79$) are made via the *r*-process.

5.5 Mass transfer in binaries

- For stars around $1\,\text{M}_\odot$ about half of all stars are in binary systems.

- The **Roche potential** of a binary system depends only on the separation between the two bodies. The surface of a start follows an equipotential line and is therefore distorted in a binary system, if the sizes are comparable to the separation of the stars.

- If a star expands (e.g. becomes a red giant) it may cause matter to spill over its Roche lobe. If further the other star is a compact object, that matter will form an **accretion disk** or cause a white dwarf to grow past the Chandrasekhar mass limit and collapse. If both stars fill their Roche lobe we have a **contact binary**, which may end up combining. This process may explain **blue straggler stars**, which are stars in a cluster with an anomalously high mass.

5.6 Pulsars and other exotic remnants

- In 1967 Bell and Hewish observed extremely regular radio pulses about 1.3 seconds apart. They called it LGM-1 ("Littel Green Men").

- For a spinning object to be held together solely by gravity, it requires a density of

$$\rho > \frac{\omega^2}{G}, \qquad \omega = 2\pi\text{s}^{-1} \;\rightarrow \rho > 10^{11}\,\text{kg}\,\text{m}^{-1}.$$

The fastest rotating objects must therefore have nuclear densities associated with neutron stars.

- Pulsars are observed to be slowing down. The ratio of the slow-down rate to the period gives an estimate of the age T of the system (typically $T \sim 10^7$ yr):

$$\frac{\dot{P}}{P} \sim \frac{1}{T}.$$

- Pulsars with magnetic dipoles which are slightly misaligned with the rotational axis also generate magnetic fields of strength around 10^8 T, over a million times stronger than any field created by humans (50 T).
- Pulsars also come in binary systems, which allows precise study of general relativity.

6 Exosolar Planetary Systems

6.1 Detection of exosolar planets

- The **radial velocity method** measures the change in radial velocity of stars (via the dopplershift) due to the orbiting planets. This only gives some lower bound values though, because we don't know about the inclination i of the orbit ($i = 0°$: "face-on", $i = 90°$: "edge-on"). The mass of a planet on a circular orbit is given by

$$M_P \sin i = v_{1,\text{obs}} M_1^{2/3} \left(\frac{T}{2\pi G}\right)^{1/3}.$$

- Massive planets orbiting close to (low-mass) stars are the easiest to detect. This has a significant selection effect.
- The **occulation or transit method** works on planets who's orbital planes are perpendicular to us ($i \approx 90°$) and measures the dip in brightness as the planet passes in front of its star:

$$\frac{\Delta f}{f} \sim \left(\frac{r_P}{r_\text{star}}\right)^2$$

A second (tiny) dip occurs when the planet passes behind the sun, as the planet doesn't reflect anymore. This method paired with the radial velocity method allows for very precise measurements. Though, this method also favors close-in large planets.
- Further, if the planet has an atmosphere, we can measure what frequencies get absorbed and therefore deduce the atmosphere's composition (**transit absorption spectroscopy**). We can also deduce whether the planets and its star's spins are aligned, by measuring the dopplershift of the blocked light, as the planet enters and exits the disc of the star.
- Planets are common, there is an average of about one planet per star. But most exoplanets are measured to have very high eccentricity, contrary to our solar system. No obvious selection effects are known.
- Finally we can detect planets with gravitational lensing (not nearly as important). If light passes by a massive object at distance b, the beam is deflected by twice the Schwarzschild radius R_S to b:

$$\alpha = \frac{4GM}{c^2 b} = \frac{2R_S}{b} \qquad \text{(thin lens approximation)}$$

- If the light comes from exactly behind the gravitational lens, it will appear as a ring called an **Einstein ring** with an angular radius of

$$\theta_E = \left(\frac{4GM}{c^2}\frac{D_\text{LS}}{D_\text{OL}D_\text{OS}}\right)^{1/2}$$

- If it is slightly displaced by an angle β we will see two images at angle $\theta_\pm$ from the lens:

$$\theta_\pm = \frac{1}{2}\left[\beta \pm (\beta^2 + 4\theta_E^2)^{1/2}\right]$$

- Gravitational lensing also amplifies images, allowing us to measure the change in brightness. If a planet orbits a gravitationally amplifying star, it might contribute and add a second brightness peak.

6.2 The heating of planets and satellites

- Planets (and moons) do not produce significant amounts of energy on their own. They radiate by reflecting light from their star and by absorbing radiation and re-emitting as thermal radiation.
- The fraction of radiation that will be reflected back into space is called **albedo** a (~ 0.30 for Earth).
- A planet at distance d will reach equilibrium temperature T_eq such that it black-body emission equals the absorption of energy from its star (with radius R_S and temp T_S):

$$T_\text{eq} = (1-a)^{1/4}\sqrt{\frac{R_S}{2d}}T_S.$$

- For planets with atmosphere, particles in the exosphere may escape it if their velocities are grater than the escape velocity. A crude rule is that the atmosphere will be gone after 4 billion years if

$$v_\text{rms} > \frac{1}{6}v_\text{esc}.$$

Note that $v_\text{rms} \propto m^{-1/2}$, so massive particles be least likely to be lost (CO_2 won't be lost, H will).
- If the atmosphere is transparent to (short wavelength) sunlight and opaque to (long wavelength) thermal emission of the planet, the surface temperature T_S is raised by

$$T_S = \sqrt[4]{2}T_\text{eq}.$$

- To force a radiation flux through an opaque atmoshpere of optical depth τ, we need a surface temperature of

$$T_S \sim \left(1 + \frac{3}{4}\tau\right)^{1/4} T_\text{eq},$$

which for earth is around $T_S \sim 1.1 T_\text{eq}$, and for Venus (CO_2 rich!) $T_S \sim 2T_\text{eq}$.
- Tidal forces arise within an extended mass in the presence of an inhomogeneous gravitational field whose strength varies through the volume of the extended mass. A moon around a planet compresses the planets poles and extends its equator (Earth: ~ 10 cm, Moon: ~ 20 m).
- If an objects rotational period doesn't match its orbital period, the tidal distortion won't be constant in the objects rotating frame. Further, friction within the object will ensure that the distortion is misaligned, either "ahead" if it spins faster, or "behind" if it spins slower. This misaligned bulge will induce a tidal torque and slow it down/speed it up (Earth: 1.6 ms per century). Once the orbital and rotational periods are synchronized, the two objects are **tidally locked**.
- If the tidal forces distorting the object become larger than its self-gravity, then it will disrupt the object. This happens once its radius r becomes:

$$r < 2^{1/3}\left(\frac{\rho_P}{\rho_m}\right)^{1/3} R_p.$$

- The changing deformations due to tidal effects may cause significant internal heating.

7 The Milky Way Galaxy

7.1 Galactic structure

- Dust in the Galaxy blocks most of the visible light, preventing visual observation of distant stars.
- The Milky Way Galaxy is a highly-flattened disk with its center around (8.3 ± 0.3) kpc from our Sun. The Sun orbits with a circular speed of $(240 \pm 20)\,\text{km}\,\text{s}^{-1}$ and has an orbital period of 2×10^8 years. The alignment of the angular momentum vectors of the discs material minimizes the kinetic energy.
- The mass distribution of the Galactic disk falls exponentially both radially and with height:

$$\rho(r,z) = \rho_0 \left[\exp\left(-\frac{r}{r_d}\right)\right]\left[\exp\left(-\frac{|z|}{h_d}\right)\right],$$

where the radial scale length is around $r_d \approx (2.5 \pm 0.5)$ kpc, and the scale height around $h_d \approx 0.13 - 0.4$kpc.
- 90 % of the mass of the disk is from stars, 10 % from gas.
- The spiral arms are density waves that are most prominent at short wavelengths. They do not orbit at the same speed as the stars.
- The inner part of the Galaxy has a spheroidal stellar component called the **bulge**, who's density falls off from the center

$$\rho(r) \propto r^{-3}.$$

About 10 % of the stellar mass is in the bulge.
- A diffuse stellar and gas halo (**Population II**) follows a similar profile and extends to about 50 kpc and contains 1 % of the Galaxy's mass. Most prominent are its **globular clusters**.
- A stellar population is a set of stars with similar metallicities, kinematics and age distributions. Three such populations are:
  · *Population I*: Disk stars, $\frac{Z}{Z_\odot} \in [0.3, 3]$, ages $0 - 10$ Gyr,
  · *Population II*: Halo stars, $\frac{Z}{Z_\odot} \in [0.3, 3]$, old,
  · *Population III*: (postulated) first generation stars, $Z \sim 0$.
- By tracking the motion of the star S2 orbiting close to the center of the Galaxy, the central object's mass has been determined to be $3 \times 10^6\,\text{M}_\odot$. As S2 comes so close, this has to be a black hole.
- The stellar number density in the central pc^3 is around 10^7 times higher than our stellar neighborhood.

7.2 The interstellar medium (ISM)

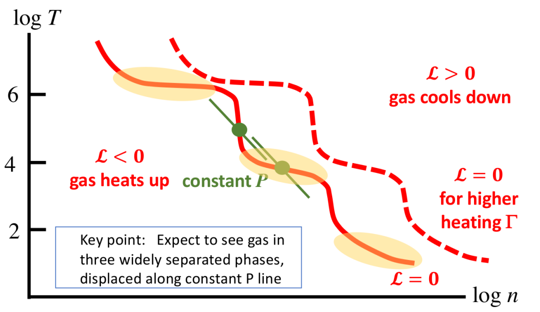
- The ISM has a very wide range of densities n ($10^{-3} - 10^3\,\text{cm}^{-3}$) and temperatures T ($10 - 10^7$ K). The pressure nT is more or less constant though.
- The ISM is mostly Hydrogen, 26 % Helium and 2 % "metals", half of which is in the form of dust (0.001-1µm, "smoke"), mostly Fe, Si, C and H_2O, CO_2 ices produces in atmospheres of very cool giants and supernovae.
- The light attenuation factor $E(\lambda)$ (for $1000\,\text{Å} < \lambda < 1\,\mu\text{m}$) increases dramatically for short wavelengths:

$$\log E(\lambda) = \frac{\lambda_0}{\lambda}\log E(\lambda_0).$$

The Galactic center is therefore best studied in infrared.
- The ISM gets heated by Supernova explosions, radiation from hot stars ($T > 20\,000$ K), mechanical shock waves, cosmic rays.
- Cooling happens through collisional excitation of quantum states that then radiatively de-excite. A gas at 10-100K is able to excite molecular rotational levels (meV), and at 10^4 K it can also excite electron orbital levels (eV). The total excitation rate per volume will go as the square of the density.
- The heating rate Γ depends on the location, ionization state, dust content etc. and the cooling rate Λ depends on the number density n, temperature T and composition (Z). For optically thin systems in equilibrium we have

$$\mathcal{L} = \Lambda - \Gamma = 0.$$

- $\mathcal{L} = 0$ defines a line in the $T - p$ diagram, above which gas cools, and and below which gas heats up. For temperatures with associated energies close to a quantum excitation transitions, changes in density n can easily be compensated by small changes in temperature T. Far away from quantum transitions, we need large changes in T. This creates several "plateaus" of the $\mathcal{L} = 0$ line (for high $T \sim 10^7$ K thermal bremsstrahlung takes over).
- The plateaus of $\mathcal{L} = 0$ are stable regions, as perturbations maintain the pressure balance. Increase in temperature on a plateau lead to cooling where as on a steep part it leads to further heating, and vice versa.



7.3 Gravitational instability

- We use Gauss's theorem, mass conservation and ignore viscous forces to obtain the **equations of fluid dynamics**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \text{(continuity equation)}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \vec{F} \qquad \text{(Euler's Equation)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{\rho^\gamma}\right) = 0 \qquad \text{(thermal equation)}$$

- A uniform medium is gravitationally unstable, if the perturbations are on too long length scales. The perturbed continuity and Poisson equations are given by

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 = 0, \qquad \left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 - 4\pi G\rho_0\right)\rho_1 = 0.$$

Note that for the Poisson equation, if gravity is neglected we get the normal acoustic wave equation, and if pressure is neglected we get an exponential collapse. Which one is relevant is given by the Fourier modes Δ_k of the perturbation:

$$\frac{\partial^2 \Delta_k}{\partial t^2} = \left\{4\pi G\rho_0 - k^2 c_s^2\right\}\Delta_k = \left\{k_J^2 - k^2\right\} c_s^2 \Delta_k,$$

which has a solution $\sim e^{i\omega t}$, with

$$\omega^2 = c_S^2\left(k^2 - k_J^2\right). \qquad \lambda_J = \frac{c_S}{\sqrt{\pi G\rho_0}}$$

is the **Jeans length**, which is the critical wavelength:

$$\lambda > \lambda_J \quad \rightarrow \quad \omega \in i\mathbb{R}, \text{ exponential collapse}$$
$$\lambda < \lambda_J \quad \rightarrow \quad \omega \in \mathbb{R}, \text{ acoustic oscillations}$$

- This also gives us the **Jeans's mass** $M_J \sim \rho_0 \lambda_J^3$ which is the minimal mass of objects created from gravitational collapse.

7.4 Star-formation and planet-formation

- The **initial mass function** (i.m.f.) describes the stellar mass distribution

$$\mathrm{d}N = \varphi(m)\,\mathrm{d}m, \qquad \varphi(m) \propto m^{-2.35},$$

which bends over for lower masses. Note that most of the mass is from low-mass stars, but luminosity scales with M^3, most of the light is from high masses.

- During the formation of a star, a gas increases in density by a factor 10^{22} This creates three main problems:
- The first problem is that enormous amounts of potential energy have to be released (fast). Once the object becomes optically thick, the release of radiative energy is greatly hindered. Solution to this might be dust grains, which absorb higher frequency radiation and emit them at lower frequencies, to which the object is less opaque. These dust grains do not survive above 1000 K though.
- Secondly, the star has to get rid of a lot of angular momentum as it contracts (if it didn't it would get ripped apart by centrifugal forces).
- Finally, a star has to remove any potential magnetic fields, as these also get amplified with the collapse. The formation of an accretion disk might solve the latter two problems.
- Planets form from these rotating accretion disks of gas and dust within a few million years. They have a very different chemical composition from their parent star, mostly made from "metals".
- Planets start as dust (µm) and grow by a factor of 10^{12} in radius. We differentiate three main stages.
- The first stage is condensation (µm → cm) and it explains the varying chemical species, as different materials condensate differently. The vapor pressure P for a given surface temperature T is given by August's Law:

$$\ln P = -\frac{\lambda}{RT} + c,$$

with each species having different λ. Planets closer to the sun have higher surface temperatures, therefore are made of metals (Al, Fe, Ni), whereas in the outer solar system planets are made from volatile ices of water, ammonia and methane.

- As these dust grains grow, their surface area to volume ratio decreases and mass increases, so gravity takes over from condensation (cm → 100 m). Gravitational collapse is given by Jeans's analysis for a 2-d disk. If we now neglect pressure we get collapse below a maximal perturbation length scale and mass:

$$M_{\max} \sim \Sigma_0 \lambda^2 = 16\pi^4 G^2 \Sigma_0^3 \Omega^{-4}$$

If pressure cannot be neglected the speed of sound has to reduced greatly for gravitational collapse to occur. This is achieved by the growing dust grains, and once they have reduced the speed of sound sufficiently the disk becomes gravitationally unstable.

- Finally, radiation pressure from the star removes any remaining gas. The final growth of the planets happens (100 m → 1000 km) happens through collisions between planetessimals and three-body interactions ejecting objects. Through collisions, planets can also change their positions in the solar system by losing/gaining angular momentum.
- The creation of giant planets is also though to be a consequence of a "direct collapse" of the gaseous accretion disk due to a substantial density inhomogeneity.

8 Galactic Dynamics

8.1 Stellar dynamics of self-gravitating systems

- The **Virial theorem** for systems of stars, in which the inward pull of gravity is balanced by the motions of the particles, states that the (average) total gravitational potential plus twice the average kinetic energy of the system is zero:

$$E_G + 2T = 0.$$

- Gravitational system differ from ideal gases in that the particles interact over long distances, so the force acting on a star is determined by the overall structure of the galaxy. The approximate number of star-star interactions needed, before their effects become significant is negligible for typical galaxies ($N \sim 10^{11}$):

$$n \sim \frac{1}{8\pi}\frac{N}{\ln N}$$

We therefore consider a "gas" of stars to be collisionless.

- Globular clusters on the other hand have way fewer stars ($N \sim 10^5$), so they are subject to evaporation, mass segregation and core collapse.
- Stars in a collisionless system will still exchange energy through changes in the gravitational potential $\Phi(\vec{x}, t)$ (**violent relaxation**). Their specific total energy $E = E_{\text{kin}} + E_{\text{pot}}$ is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial \Phi}{\partial t}.$$

In the equilibrium state $\frac{\partial\Phi}{\partial t} = 0$ and we have Kepler dynamics. Note that the total specific energy $\sum E$ stays constant.

- All stars will end up with the same average specific energy, independent of their individual masses. In contrast, the collisional relaxation in ideal gases, the particles end up with the same *kinetic* energy (specific energy varies with m^{-1}).

8.3 Rotation curves and dark matter haloes

- Using normal Newtonian mechanics, we expect the radial dependence of the circular velocity $v_c(r)$ (the **rotation curve**), for the flattened exponential mass distribution of a galaxy $\Sigma(R) = \Sigma_0 e^{r/r_d}$ to initially rise and then exponentially fall to zero (messy function containing modified Bessel functions).
- What we observe though, is that the velocity curve initially rises as expected but then stays constant. This implies a dark halo of dark matter around the galaxy, which we cannot detect. This dark matter is estimated to have a mass of $10^{12}\,\mathrm{M}_\odot$, which is 16 times more than the observable mass (6-7 $\times 10^{10}\,\mathrm{M}_\odot$).
- There is also other dynamic evidence of dark matter, for example estimating the mass of galaxy clusters using the Virial theorem and the light-to-mass ratio.
- This does not necessarily mean that dark matter is non-baryonic, some voices even say that classical dynamics may be wrong (g for sun around galaxy is 10^7 times smaller that for earth around sun). But other phenomena also support the idea of dark matter being non-baryonic.

9 Active Galactic Nuclei (AGN)

9.1 AGN phenomenology

- Radio sourced that actually were very distant stars with high redshift are called **quasi-stellar radio source** or **quasars**. These were mostly found at the core of so called **Seyfert Galaxies**. They have luminosities up to 10^{47} erg/s (1000 times more than Milky Way) and had variations on timescales down to a few hours. This can only be caused by very small orbits, therefore requiring the energetics of a supermassive black hole.
- **Type 1 AGN** (some) show broad emission lines from permitted transitions and high ionization species in dense gas moving with high bulk velocities (1000-20000 km s^{-1}).
- **Type 2 AGN** (most) show narrow emission lines from forbidden transitions of normal density gas moving with normal velocities (100 km s^{-1}).
- Most type 1 AGN are strong compact continuum sources at visible and ultraviolet wavelengths due to thermal emission from a hot accretion disk.
- AGNs are rare X-ray and γ-ray sources, primarily due to comptonization of visible/ultraviolet photons by an extremely hot corona.
- Infrared radiation is due to a thick torus of dust mixed with gaseous material absorbing radiation and re-emitting it.
- Type 1 and Type 2 AGNs are actually thought to be the same objects, just viewed from different angles, as some type 2s also show weak, highly polarized narrow emission lines.
- Few (10^{-3}) AGNs are **strong radio sources** ("radio loud"). The radio radiation mostly comes from the central engine or two huge lobes ejecting highly collimated jets of relativistic particles. These are usually found in massive elliptical galaxies that have stopped forming stars, which are different in mass and spin of the black hole, have higher accretion rate and the jet can more easily propagate through the galaxy.

9.2 Accretion disks

- If the rotational flow is super sonic $v_\varphi \gg c_s$, the disk is thin.
- Viscous torques in an accretion disk acts to move mass inwards and angular momentum outwards.
- In the steady state, where the disk gains mass at the same rate as it deposits it in the core, the radial mass flow is constant $\dot{m} = $ const. and therefore losing gravitational potential energy. One half of which goes to increase the kinetic energy of the disc and the other half increases the luminosity. For non-rotating black holes this is given by

$$L_{\text{nrBH}} = \frac{1}{12}\dot{m}c^2.$$

- The emission spectrum scales as $I_\nu \propto \nu^{1/3}$.

9.3 Eddington Luminosity

- As the radiation force acts mostly on electrons (through Thompson scattering) and gravitational force is dominated by protons, accretion flow can only be maintained if $F_{\text{grav}} > F_{\text{rad}}$. This limits the luminosity (**Eddington luminosity**):

$$L_{\text{edd}} = \frac{4\pi c G M m_p}{\sigma_T}.$$

- The luminosity of an accretion disk is determined by the mass accretion rate and the radiative efficiency is primarily dependent on the spin of the black hole. We therefore need supermassive black holes to attain the very high luminosities that are observed.
- The mass of a central black hole will grow exponentially with an e-folding time of $\tau_S \approx 25$ Myr (**Salpeter time**).

9.4 Effects due to high velocities

- Superluminal motion is a geometrical effect that makes objects moving almost along the line of sight at high velocities appear to be moving transversally at over the speed of light.
- Relativistic beaming is the effect that radiation from a rapidly moving source is beamed in the direction of its motion.
- The effects of time dialation and Doppler shift can substantially affect the specific intensity (i.e surface brightness).

10 The expanding Universe

10.1 Basic cosmological observations

- Hubble showed in 1929 that the redshift z is proportional (**Hubble's constant** $H_0 \approx 70\,\mathrm{km s^{-1} Mpc^{-1}}$) to the distance:

$$(1+z) = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{\nu_{\text{em}}}{\nu_{\text{obs}}}, \qquad z = \frac{H_0}{c}d, \qquad v = H_0 d$$

- Galaxies are receding away from us, any physical measurement would show that distance increases over time. This is the signature of a uniformly expanding medium, in which none is the center of expansion.
- Locally we of course see some scattering due to gravitational perturbations inducing **peculiar velocities**.

- The CMB is an almost perfect black-body (2.726 K) which dominates the energy density of radiation. It has the same surface brightness down to the 10^{-3} level, and once the earths motion and the light from the galaxy is subtracted, fluctuations become apparent at only 10^{-5}
- The universe appears isotropic and homogeneous on larger and larger scales. The **cosmological principle** states:

> "*At any epoch, the Universe appears the same to all observers, regardless of their individual locations.*"

- A **fundamental observer** is someone who observes an isotropic CMB, i.e. has no peculiar velocity.
- Unexplained cosmological observations are:
  · Abundance of ^{4}He,
  · Maximum age of objects in the Universe,
  · Existence of large scale structures,
  · Evidence of cosmic evolution.

10.2 The Robertson-Walker metric

- In order to do physics "over there, long ago" based on observations now "here and now", we need a metric based on fundamental observers. This is the **Robertson-Walker metric** based on polar comoving coordinates $(\omega, \theta, \varphi)$ or (r, θ, φ):

$$\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2 - R^2(\tau)\left\{\mathrm{d}\omega^2 + s^2(\omega)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right\}$$

$$\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2 - R^2(\tau)\left\{\frac{\mathrm{d}r^2}{1-kr^2/A^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right\},$$

where $R(\tau)$ is the (dimensionless) cosmic scale factor and $s(\omega)$ accounts for possible curvature.

- For the $(\omega, \theta, \varphi)$ coordinates, this curvature is fully incorporated in the angles, meaning radii ω always add straightforwardly.
- For the (r, θ, φ) coordinates the curvature affects all three coordinates, where $k = +1$ is for positively curved, $k = -1$ for negatively curved, and $k = 0$ for non-curved (Euclidean) spacetime.

10.3 Some consequences of the Robertson-Walker metric

- The cosmological redshift is a measure of the factor by which the expansion factor R has changed between τ_{em} and τ_{obs}:

$$(1+z) = \frac{R(\tau_{\mathrm{obs}})}{R(\tau_{\mathrm{em}})}$$

Note that we have two versions of redshift now: Doppler shift due to recessional (peculiar) velocities and a cosmological effect due to expansion.

- Hubble's constant represents the normalized expansion rate of the universe and has unit of 1/time:

$$H = \frac{\dot{R}}{R}.$$

- The brightness of light sources is decreased by two factors of $(1+z)$, one from the energy change of each photon, and one from the change in their emission rate.

$$f = \frac{L}{4\pi\underbrace{R_0^2 s^2(\omega_i)(1+z)^2}_{\text{luminosity distance } D_L^2}}$$

- Gravitationally bound structures maintain their size because their densities are so much higher and therefore are governed by "local" physics. A consequence of this is that an object with fixed physical size at increasing distances first gets smaller in angular size, but then starts getting larger again, because it is so far in the past, when the universe was still smaller.
- The surface brightness I, I_ν is therefore decreased by four factors $(1+z)$ (not constant anymore!), two because $\mathrm{d}A_2$ is now physically bigger than $\mathrm{d}A_1$, one because photons have less energy now, and one from the decreased rate of their emission:

$$I_{\mathrm{obs}} = \frac{I_{\mathrm{em}}}{(1+z)^4}, \qquad I_{\nu,\mathrm{obs}} = \frac{I_{\nu,\mathrm{em}}(\nu(1+z))}{(1+z)^3}.$$

- The black body radiation field at some redshift z changes to that of a decreased temperature of $T/(1+z)$, therefore

$$RT = \text{constant}.$$

The 2.7 K CMB therefore originated from the very early universe, when it was very much hotter and smaller (denser) and thermodynamic equilibrium could more easily be reached.

10.4 The Friedmann equation

- The **Friedmann equation** derived from GR field equations is

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \frac{kc^2}{A^2} + \frac{\Lambda}{3}R^2,$$

with R the scale factor, ρ the density as a function of R and Λ Einstein's cosmological constant, a density component independent of R.

- This equation can be derived with a purely Newtonian approach by looking at the EOM of a point A a distance $x = R\omega_O A$ away from an observer O and assuming pressure-less matter, i.e. $\rho \propto R^{-3}$.
- The change in density as the universe expands can be given in several ways.
  · For normal (non-relativistic) matter we said

$$\rho_m \propto R^{-3}.$$

  · For radiation and other relativistic matter we have

$$\rho_\gamma \propto R^{-4}.$$

  · For a false vacuum ρ is constant:

$$\rho_\Lambda \propto R^0.$$

  · And finally some "cosmic strings" with constant density no matter their size

$$\rho_\circ \propto R^{-2}.$$

These can also be derived from thermodynamical equations of states linking p to ρ. All of these could be used with the Newtonian analysis if we use the "active density" $\rho' = \rho + 3p/c^2$.

10.5 Asymptotic solutions to the Friedmann equation

- For a pressure-less, matter-dominated universe with negligible curvature the Friedmann equation is solved as follows, and with the elapsed time since the Big Bang ($R = 0$):

$$\frac{R}{R_0} = \left(\frac{3}{2}\right)^{2/3}\left[\frac{8\pi G\rho_0}{3}\right]^{1/3}\tau^{1/3} = \left(\frac{3}{2}H_0\tau\right)^{2/3}, \qquad \tau_0 = \frac{2}{3}H_0^{-1}$$

- If the universe is dominated by radiation of other relativistic species we get

$$\frac{R}{R_0} = \left(\frac{32\pi G}{3c^2}U_0\right)^{1/4}\tau^{1/2}, \qquad \tau_0 = \frac{1}{2}H_0^{-1},$$

where U_0 is the energy density of the radiation field in the present.

- For an expansion dominated by false vacuum with constant density we get an exponentially accelerating expansion

$$\frac{R}{R_0} = \exp\left[H(\tau - \tau_0)\right], \qquad H = \sqrt{\frac{8\pi G\rho_\Lambda}{3}},$$

With Hubble's parameter H being constant. Gravity here seems to act repulsively. This may be linked to "Dark Energy".

- If the curvature term dominates we get a solution for negative curvature $k = -1$:

$$\frac{R}{R_0} = H_0\tau, \qquad \tau_0 = H_0^{-1}.$$

- We assumed that the different R dependencies evolve independently, but in reality they could change from one form into another over time. Currently the density is dominated ($\sim 70\,\%$) by a false vacuum.

10.6 The interrelation between the curvature, the density and the expansion rate

- We can define a **critical density** ρ_c and a density parameter Ω for each density component i

$$\rho_c = \frac{3H^2}{8\pi G}, \qquad \Omega_i = \frac{\rho_i}{\rho_c},$$

to rewrite the Friedmann equation into

$$\sum_i \Omega_i - \frac{kc^2}{\dot{R}^2A^2} = 1.$$

- We can see that a universe with critical density, i.e. $\Omega = \sum \Omega_i = 1$ has zero curvature.
- A decelerating universe ($\dot{R}$ decreasing), Ω is diverging.
- An accelerating universe ($\dot{R}$ increasing), Ω is converging towards unity.
- Currently Ω is close to unity, meaning the curvature is negligible on the characteristic scale of c/H (scale of the universe).

11 Thermal history of the Big Bang

11.1 Matter and radiation

- For all models of the expanding universe, there was a point in time, when R was very small and the density and pressures were extremely high. During that time the ratio of particles to photons is constant. Once thermodynamic equilibrium is reached though, matter and radiation evolve independently:

Radiation:	$n_\gamma \propto R^{-3}$	$\rho_\gamma \propto R^{-4}$	$T \propto R^{-1}$
Matter:	$n_m \propto R^{-3}$	$\rho_m \propto R^{-3}$	$T \propto R^{-2}$

- Particles species of mass m_p have a threshold temperature T_p give by

$$kT_p = m_pc^2.$$

- Above T_p particles are relativistic therefore contribute to ρ_γ and particle/anti-particle pairs are being created and annihilated.
- Below T_p particles are non-relativistic, contribute to ρ_m and all particle/anti-particle pair annihilate each other. As we have mostly matter now, there must have been slightly more particles than anti-particles (**Baryogenesis**).

11.2 Reaction rates and the maintenance of equilibrium

- For a reaction in equilibrium the reaction rate $n\sigma v$ and its timescale $(n\sigma v)^{-1}$ in both ways are equal.
- For an expanding universe the equilibrium configuration continuously changes due to changes in temperature and density. This equilibrium can only be maintained as long as the reaction timescale is shorter than the age of the universe, otherwise we get **Freeze-out**. The abundances of the reagents after the freeze-out, will be set by the equilibrium value, when it was lost.
- Freeze-out is why we still have such high densities of neutrinos and anti-neutrinos.
- The neutron/proton ratio has also been frozen since $\tau \sim 1\,$s.

11.3 Big Bang Nucleosynthesis (BBNS)

- Free neutrons are unstable to β-decay. Bound ones are stable. But until $\tau > 100\,$s, when $T < 10^9\,$K, γ rays breaks deuterium ^{2}He apart again (**Deuterium bottleneck**). After that, a series of reactions happen to create ^{4}He, with a final abundance of $Y = 0.24$. This calculated value depends on the time-temperature relation of the universe, the number density of neutrinos and our understanding of the weak interaction.

11.4 Recombination and the Last Scattering Surface of the CMB

- In the early universe baryonic matter was a fully ionized plasma, which hindered the propagation of light (opaque). Once everything started to cool down and the protons and electrons combined into hydrogen (**recombination**), the mean free path for photons increased dramatically (Thompson scattering for bound electrons is vastly smaller), so the universe became transparent.

- The last scattering surface describes the distance at which each photon was last time Thompson scattered. It is approximately a Gaussian centered at $z = 1065$ with $\sigma_z \sim 80$, so most photons were last scatter in a very narrow interval between $950 < z < 1100$, which is around 370000 years after the Big Bang.
- By looking at the inhomogeneities of the CMB we see the structure of the early universe.
- If we ever manage to look at the last scattering surface of neutrinos, we could see the universe 1 sec after the Big Bang.

12 The formation of structure

12.1 Linear density fluctuations

- We will discuss the density fluctuations $\Delta(\vec{x})$ of the CMB as the Fourier sum of different modes $\Delta_k(\vec{k})$. These modes are observed to be random, therefore the probability distribution of the over-density $\Delta(\vec{x})$ is given by a Gaussian. Further, we expect the power spectrum $\Delta_k^2(k)$ to be given by a power-law in k:
$$\Delta_k^2 \propto k^n.$$
- The Jeans-Lifshitz equation for an expanding universe adds a "damping term" to the equation for gravitational instability, which again has oscillating solutions for modes below the Jeans length.
- In a universe dominated by matter, the growth above the Jeans length is simply linear (not exponential!):
$$\Delta_k = \Delta_{k,0} \left(\frac{R}{R_0} \right).$$
Can think of this as gravity becoming weaker and weaker because the particles are moving further and further apart.
- In a universe dominated by radiation, the growth above the Jeans length is quadratic:
$$\Delta_k = \Delta_{k,0} \left(\frac{R}{R_0} \right)^2.$$
- An empty universe dominated by its curvature shows no growth at all.
- In the beginning, when both matter and radiation are equally dominant, the speed of sound is very close to the speed of light. Jeans length is therefore huge, essentially on the horizon scale (distance light can travel in the age of the universe). This means that no structure will grow during this time.
- Photon diffusion in the matter density oscillations before recombination, will cause there fluctuations to be dampened out, causing all structure below the so called **Silk mass** to be washed out (**Silk damping**). This Silk mass is way larger than the galaxies we observe, we would therefore not expect galaxies to form. But due to non-interacting dark matter they did anyway.
- After the universe becomes mass dominated and recombination occurs, the Jeans mass will drop down to $10^6\,\mathrm{M}_\odot$.
- As dark matter doesn't interact (except gravity), it has no pressure effects and no Jeans mass. Its fluctuations also don't grow, which is why we call it **Cold** dark matter (instead of Hot).
- The Meszaros effect does damp/oscillate some dark matter fluctuations, giving a characteristic distribution of scales in the universe.
- CMB fluctuations with size of around 1 angular degree show the biggest growth. Smaller objects wiggle due to phasing effects and the smallest ones have been suppressed due to silk damping and the thickness of the LSS.

12.2 Non-linear collapse

- Until now we assumed that the density perturbations were small, but these will grow and behave non-linearly. Their behavior is best studied in large scale numerical simulations.
- Violent relaxation will establish an equilibrium self-gravitating object satisfying the Virial condition. The final over-density will be a factor of 200 relative to the surrounding universe. These objects are massive enough that they will remain stable and not be largely affected by the expanding universe.
- Three-dimensional self-gravitating dark matter structures with densities >200 times the average universe are called **dark matter haloes**.
- Silk damping will have erased all structure on galactic scales. The dark matter fluctuations have stayed though. Once the radiation pressure is removed, baryons will quickly fall into the gravitational potential wells of the dark matter fluctuations.
- Galactic scale cooling of the dark matter haloes (baryons loose energy and sink down into the potential wells) and conservation of angular momentum is what form a spinning disk, what we call a galaxy.
- The inability of dark matter to cool is what leads to the concentration of baryons in the center of galaxies.

12.3 Summary: Concordance cosmology

- From precision measurements of the CMB temperature fluctuations, analysis of light element abundances, measurements of the local expansion rate, analysis of the redshift-distance relation for $\ddot{R}(t)$ and estimates of the matter density, we have developed the ΛCDM-model of the universe.
- We require 85 % of the matter to be non-baryonic (Cold **Dark Matter**).
- We require 70 % of the density of the universe to be in the form of something that mimics the effects of a false vacuum (constant density as the universe expands). This **Dark Energy** causes the universe expansion to accelerate.