1 Klassische Mechanik

Translation in eine Richtung			Rotation um feste Achse		
Ort	x	m	Winkel	φ	rad
Gesch.	$\boldsymbol{v} = \dot{x}$	m/s	Winkelgeschw.	$\boldsymbol{\omega} = \dot{\varphi}$	rad/s
Besch.	$\boldsymbol{a} = \ddot{x}$	m/s^2	Winkelbeschl.	$\boldsymbol{\alpha} = \ddot{\varphi}$	rad/^{2}
Masse	m	kg	Trägheitsmoment	$I = \int r_\perp^2 dm$	kg m^2
Impuls	$\boldsymbol{p} = m\boldsymbol{v}$	Ns	Drehimpuls	$\boldsymbol{L} = I\boldsymbol{\omega}$	Nms
Kraft	$\boldsymbol{F} = \dot{\boldsymbol{p}} \overset{m=const}{=} m\boldsymbol{a}$	N	Drehmoment	$\boldsymbol{M} = \dot{\boldsymbol{L}} \overset{I=const}{=} I\boldsymbol{\alpha}$	N m
Arbeit	$W = \int \boldsymbol{F} \cdot d\boldsymbol{r}$	J	Dreharbeit	$W = \int \boldsymbol{M} \cdot d\boldsymbol{\varphi}$	J
Kinetische E.	$T = \frac{1}{2}m\boldsymbol{v}^2 = \frac{\boldsymbol{p}^2}{2m}$	J	Rotationsenergie	$T = \frac{1}{2}I\boldsymbol{\omega}^2 = \frac{\boldsymbol{L}^2}{2I}$	J
Leistung	$P = \dot{W} = \boldsymbol{F} \cdot \boldsymbol{v}$	W	Drehleistung	$P = \boldsymbol{M} \cdot \boldsymbol{\omega}$	W

Reduced Mass $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$

Two-body problem $\boldsymbol{F} = \mu \ddot{\boldsymbol{r}}_{21}, \quad I = \mu r_{21}^2$

2 Electromagnetic Waves

Vacuum permittivity $\varepsilon_0 = 8.8542 \times 10^{-12}\ \frac{\text{As}}{\text{Vm}}$, vacuum permeability $\mu_0 = 1.2566 \times 10^{-6}\ \frac{\text{Vs}}{\text{Am}}$, speed of light in vacuum $c = \frac{1}{\sqrt{\varepsilon_0 \mu 0}} = 3 \times 10^8\ \frac{\text{m}}{\text{s}}$

Lorentz force $\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

2.1 Maxwell Equations

Microscopic	Integral equations	Differential equations
Gauss's law	$\oiint_{\partial\Omega} \boldsymbol{E}\mathrm{d}\boldsymbol{n} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \mathrm{d}V$	$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\oiint_{\partial\Omega} \boldsymbol{B}\mathrm{d}\boldsymbol{n} = 0$	$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$
Faraday's law	$\oint_{\partial A} \boldsymbol{E}\mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t} \iint_{A} \boldsymbol{B}\mathrm{d}\boldsymbol{n}$	$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$
Maxwell-Ampere'slaw	$\oint_{\partial A} \boldsymbol{B}\mathrm{d}\boldsymbol{s} = \mu_0 \iint_{A} \boldsymbol{J}\mathrm{d}n + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \iint_{A} \boldsymbol{E}\mathrm{d}\boldsymbol{n}$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$

Macroscopic	Integral equations	Differential equations
Gauss's law	$\oiint_{\partial\Omega} \boldsymbol{D}\mathrm{d}\boldsymbol{n} = \iiint_{\Omega} \rho_{\text{free}} \mathrm{d}V$	$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}}$
Gauss's law for magnetism	$\oiint_{\partial\Omega} \boldsymbol{B}\mathrm{d}\boldsymbol{n} = 0$	$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$
Faraday's law	$\oint_{\partial A} \boldsymbol{E}\mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t} \iint_{A} \boldsymbol{B}\mathrm{d}\boldsymbol{n}$	$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$
Maxwell-Ampere'slaw	$\oint_{\partial A} \boldsymbol{H}\mathrm{d}\boldsymbol{s} = \iint_{A} \boldsymbol{J}_{\text{free}}\mathrm{d}\boldsymbol{n} + \frac{\partial}{\partial t} \iint_{A} \boldsymbol{D}\mathrm{d}\boldsymbol{n}$	$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t}$

Displacement field $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ (in vacuum $\boldsymbol{P} = 0$, in linear optics $\boldsymbol{D} = \varepsilon\varepsilon_0 \boldsymbol{E}$)

Magnetization field $\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$ (in vacuum $\boldsymbol{M} = 0$, in linear optics $\boldsymbol{H} = \frac{1}{\mu\mu_0}\boldsymbol{B}$)

Induced electric polarization $\boldsymbol{P} = \chi\varepsilon_0 \boldsymbol{E}$ (for small fields), satisfies $\operatorname{div} \boldsymbol{P} = -\rho_{\text{bound}}$

Induced magnetization field $\boldsymbol{M} = \chi_m \frac{1}{\mu_0}\boldsymbol{B}$ (for small fields), satisfies $\operatorname{rot} \boldsymbol{M} = \boldsymbol{J}_{\text{bound}} - \frac{\partial \boldsymbol{P}}{\partial t}$

ρ_{free} is the free charge density, $\boldsymbol{J}_{\text{free}}$ is the free current density, χ is the electric susceptibility and χ_m is the magnetic susceptibility.

Refractive index $n = \sqrt{\varepsilon\mu} = \frac{c}{v_p}$

2.2 Wave Equation

Homogenous dielectric material without free charges or electric currents

$$\triangle \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} = \mu_0 \frac{\partial^2}{\partial t^2}\boldsymbol{P}$$

In Vacuum $\triangle \boldsymbol{E} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} = 0 \qquad \triangle \boldsymbol{H} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\boldsymbol{H} = 0$

Monochromatic solution to the wave equation

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_0^+ e^{i(\omega t - kz)} + \boldsymbol{E}_0^- e^{i(\omega t + kz)}, \qquad \boldsymbol{H}(z,t) = \boldsymbol{H}_0^+ e^{i(\omega t - kz)} + \boldsymbol{H}_0^- e^{i(\omega t + kz)}$$

2.3 Transverse Elecromagnetic Wave (TEM)

$$\boldsymbol{E} \perp \boldsymbol{H}, \quad \boldsymbol{k} \perp \boldsymbol{E}, \quad \boldsymbol{k} \perp \boldsymbol{H}$$

$$H_0^{\pm} = \pm\frac{1}{Z}E_0^{\pm}$$

vacuum: $Z = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{c}{\mu_0} = \frac{1}{c\varepsilon_0} = 377\,\Omega$, medium: $Z = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}} \overset{\mu=1}{=} \frac{Z_0}{n}$

2.4 Energy Density, Poynting Vector and Intensity

Energy density (instantaneous) $w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) = \frac{1}{2}(\varepsilon\varepsilon_0 \boldsymbol{E}^2 + \mu\mu_0 \boldsymbol{H}^2)$

Poynting vector (instantaneous) $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ (for real fields)

Poynting vector (averaged over time) $\langle \boldsymbol{S} \rangle = \frac{1}{2}\operatorname{Re}(\boldsymbol{E}_0 \times \boldsymbol{H}_0^*)$ (for complex fields)

Intensity $I = \langle \boldsymbol{S} \rangle = c_n \langle w \rangle$, from point source: $I = \frac{P}{4\pi r^2}$

Power $P = \oiint_{\partial\Omega} \boldsymbol{S}\mathrm{d}\boldsymbol{n}$

Radiation pressure $p_{\text{radiation}} = \frac{I}{c} = \frac{\langle S \rangle}{c}$ (factor by 2 if beam is reflected)

for monochromatic plane waves with amplitudes $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, where $\boldsymbol{k}_0 = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$:

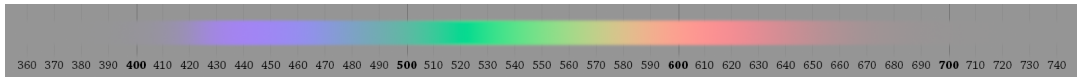
Magnetic field $\boldsymbol{H}(\boldsymbol{r},t) = \frac{\boldsymbol{k}_0}{Z} \times \boldsymbol{E}(\boldsymbol{r},t)$

Energy density (averaged over time) $\overline{w} = \frac{1}{2}\varepsilon\varepsilon_0 |\boldsymbol{E}_0|^2$

Poynting vector (instantaneous) $\boldsymbol{S}(\boldsymbol{r},t) = \frac{1}{Z}\boldsymbol{E}^2 \boldsymbol{k}_0$

Poynting vector (averaged over time) $\overline{\boldsymbol{S}} = \frac{1}{2Z}|\boldsymbol{E}_0|^2 \boldsymbol{k}_0$

Intensity $I = \frac{1}{2Z}|\boldsymbol{E}_0|^2 = \frac{1}{2}Z|\boldsymbol{H}_0|^2$, vacuum: $I = \frac{\varepsilon_0 c}{2}|\boldsymbol{E_0}|^2 = \frac{\mu_0}{2c}|\boldsymbol{E_0}|^2$



3

Optics

	Vacuum	Dispersive medium
Frequency	ν	ν
Angular frequency	$\omega = 2\pi\nu$	$\omega = 2\pi\nu$
Period	$T = \frac{1}{\nu}$	$T = \frac{1}{\nu}$
Propagation speed	$v_p = c$	$v_p = c_n = \frac{c}{n}$
Wave number	$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$	$k_n = \frac{\omega}{v_p} = \frac{\omega}{c}n = kn$
Wavelength	λ	$\lambda_n = \frac{\lambda}{n}$

Reflection $\omega_i = \omega_r = \omega_t, \quad k_{ix} = k_{rx} = k_{tx}, \quad \theta_i = \theta_r$

Snell's law $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Polarization

- Linear: $E(z,t) = E_0 \cos(\omega t - kz)$
- Right-circular: $E(z,t) = E_0 \left(\cos(\omega t - kz), \sin(\omega t - kz), 0\right)^T$ $\perp$ waves with $\Delta\varphi = \pm\frac{\pi}{2}$
- Left-circular: $E(z,t) = E_0 \left(\sin(\omega t - kz), \cos(\omega t - kz), 0\right)^T$ $\perp$ waves with $\Delta\varphi = \pm\frac{\pi}{2}$
- Elliptical: $E(z,t) = \left(E_1, E_2 e^{i\Delta\varphi}, 0\right)^2 e^{i(\omega t - k_n x)}$ $\perp$ waves with $\Delta\varphi$

Where $\boldsymbol{n}$ is a normal vector of a plane:
p-polarization: E-field lies within the plane defined by $\boldsymbol{k}$ and $\boldsymbol{n}$.
s-polarization: E-field is perpendicular to the plane defined by $\boldsymbol{k}$ and $\boldsymbol{n}$.

Fresnel equations transmitted and reflected portion of $\boldsymbol{E}$-field amplitude:

p-polarization: $t_p \equiv \frac{E_t}{E_i} = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}, \quad r_p \equiv \frac{E_r}{E_i} = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}$

s-polarization: $t_s \equiv \frac{E_t}{E_i} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}, \quad r_s \equiv \frac{E_r}{E_i} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_1\cos\theta_t} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$

using impedance vacuum: $Z_0 = 377\,\Omega$, medium: $Z = \frac{Z_0}{n}$

p-polarization: $Z_{i,p} = Z_i \cos\theta_i, \quad r_p = \frac{Z_{2,p} - Z_{1,p}}{Z_{2,p} + Z_{1,p}}, \quad t_p = \frac{Z_2}{Z_1}\frac{2Z_{1,p}}{Z_{2,p} + Z_{1,p}}, \quad t_p = \frac{Z_2}{Z_1}(1 - r_P)$

s-polarization: $Z_{i,s} = Z_i \frac{1}{\cos\theta_i}, \quad r_s = \frac{Z_{2,s} - Z_{1,s}}{Z_{2,s} + Z_{1,s}}, \quad t_p = \frac{2Z_{2,s}}{Z_{2,s} + Z_{1,s}}, \quad t_p = 1 + r_s$

For $\boldsymbol{H}$-field: $r_H = \frac{H_r}{H_i} = -r_E, \quad t_H = \frac{H_t}{H_i} = \frac{Z_1}{Z_2}t_E$

Brewster's angle $\theta_B = \arctan(n_2/n_1)$

- reflected and refracted ray stand perpendicular to each other: $\theta_1 = \theta_B$, $\theta_B + \theta_2 = 90°$
- p-polarized light is not reflected ($r_p = 0$)

Phase shifts on reflection	$\theta_i < \theta_B$	$\theta_i > \theta_B$
optically thick medium $n_1 < n_2$	$\varphi_s = \varphi_p = \pi$	$\varphi_s = \pi$, $\varphi_p = 0$
optically thin medium $n_1 > n_2$	$\varphi_s = \varphi_p = 0$	$\varphi_s = 0$, $\varphi_p = \pi$

Total internal reflection $\theta_T \geq \arcsin\frac{n_2}{n_1}$ (only when $n_1 > n_2$)

Interference $\Delta\varphi = kr_1 - kr_2 = \frac{2\pi}{\lambda}(r_1 - r_2), \quad \Delta\varphi = \begin{cases} 2m\pi & \text{constructive interference} \\ (2m+1)\pi & \text{destructive interference} \end{cases}$

Moiré $\Lambda = \frac{2\pi}{|\boldsymbol{k}_2 - \boldsymbol{k}_1|} = \frac{\lambda}{2\sin(\theta/2)}$

Fabry-Perot $R_{FP} = \frac{|E_r|^2}{|E_i|^2} = \frac{F\sin^2(k_2 d)}{1 + F\sin^2(k_2 d)}, \quad T_{FP} = \frac{|E_t|^2}{|E_i|^2} = \frac{1}{1 + F\sin^2(k_2 d)} = 1 - R_{FP}$, F-number: $F = \frac{4r^2}{(1-r^2)^2} = \frac{4R}{(1-R)^2}, \quad R = |r_{12}|^2 = |r_{21}|^2$, Resonances: $\sin^2(k_{2m}d) = 0 \Leftrightarrow k_2 md = m\pi, \quad \lambda_m = \frac{2n_2 d}{m}, \quad \nu_m = m\frac{c}{2n_2 d}$, HWFM of resonance: $\delta\nu = \frac{\Delta\nu}{f}$, free spectral range: $\Delta\nu = \nu_{m+1} - \nu_m = \frac{c}{2n_2 d}$

Finesse $f = \frac{\pi\sqrt{R}}{1-R} = \frac{\pi}{2}\sqrt{F}$

Fraunhofer Diffraction applicable when diffracted wave is observed in the far field ($d \gg b$)
Change in convention for plane wave: $E(z,t) = E_0 e^{i(\omega t - kz)}$
1-D: $T(\nu_x) = \int t(x) e^{i2\pi(\nu_x x)}\mathrm{d}x, \quad I_{\text{out}}(x) \propto \left|T\left(\frac{x}{\lambda d}\right)\right|^2, \quad \nu_x = \frac{x}{\lambda d}$
2-D: $T(\nu_x, \nu_y) = \int\int t(x,y) e^{i2\pi(\nu_x x + \nu_y y)}\mathrm{d}x\mathrm{d}y, \quad I_{\text{out}}(x,y) \approx \frac{I_{\text{in}}}{(\lambda d)^2}\left|T\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)\right|^2$

- single slit: $I(x) = I_0 \operatorname{sinc}^2\left(\frac{bx}{\lambda d}\right), \quad I_0 = \frac{I_i b^2}{\lambda d}$, angle of first minimum $\theta = \frac{\lambda}{b}$
- square aperture: $I(x,y) = I_0 \operatorname{sinc}^2\left(\frac{D_x x}{\lambda d}\right)\operatorname{sinc}^2\left(\frac{D_y y}{\lambda d}\right), \quad I_0 = \frac{I_i}{(\lambda d)^2}(D_x D_y)^2, \quad \theta_{x,y} = \frac{\lambda}{D_{x,y}}$
- round aperture: $I(x,y) = I_0\left(\frac{2J_1(\pi D\rho/\lambda d)}{\pi D\rho/\lambda d}\right)^2, \rho = \sqrt{x^2 + y^2}, \quad I_0 = \frac{I_i}{(\lambda d)^2}\left(\pi\frac{D^2}{4}\right)^2, \quad \theta \approx 1.22\frac{\lambda}{D}$
- double slits with distance a: $I(x) = I_0 \cos^2\left(\frac{\pi a}{\lambda d}x\right)\operatorname{sinc}^2\left(\frac{b}{\lambda d}x\right), \quad I_0 = \frac{4I_i b^2}{\lambda d}, \quad \theta = \frac{\lambda d}{b}$

Diffraction on a grid: $t(x) = t(x + \Lambda) = \sum_{m=-\infty}^{\infty} T_m e^{-im\frac{2\pi}{\Lambda}x}, \quad T_m = \frac{1}{\Lambda}\int_0^\Lambda t(x) e^{im\frac{2\pi}{\Lambda}x}\mathrm{d}x$
$\nu_{x,m} = m\frac{1}{\Lambda} \Leftrightarrow \theta_{x,m} \approx m\frac{\lambda}{\Lambda}, \quad I_m \propto |T_m|^2$

Pulse Dispersion

- Wave shape: $E(0,t) = A(t)e^{i\omega_0 t}$
- Fourier transform: $\tilde{E}(\omega) = \int_{\mathbb{R}} E(t) e^{-i\omega t}\mathrm{d}t$
- Envelope: $A(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\omega - \omega_0)t}\mathrm{d}\omega, \quad \tilde{A}(\omega) = \tilde{E}(\omega)$
- Pulse duration τ_p: $I(\tau_p) = \frac{1}{2}, \quad I(t) \propto |E(t)|^2$
- FWHM of $\tilde{I}$, $\Delta\omega_p$: $\tilde{I}(\Delta\omega_p) = \frac{1}{2}, \quad \tilde{I}(\omega) \propto |E(t)|^2$
- $\Delta\nu_p = \frac{\Delta\omega}{2\pi}$
- $\frac{\Delta\lambda_P}{\lambda_0} = \frac{\Delta\nu_P}{\nu_0}$
- $\tau_P' = D_\lambda L \Delta\lambda_P$

Gaussian pulse $E(t) = \underbrace{E_0 e^{-\Gamma t^2}}_{A(t)} e^{i\omega_0 t}, \quad \varphi_{\text{tot}} = \omega_0 t + \Gamma_2 t^2, \quad \omega(t) = \frac{\mathrm{d}\varphi_{\text{tot}}}{\mathrm{d}t} = \omega_0 + 2\Gamma_2$

Time-bandwidth product (Gaussian shaped pulses) $\tau_p \Delta\nu_p = \tau_p \frac{\Delta\omega_p}{2\pi} = \frac{2\ln 2}{\pi} = 0.4413$

Helmholtz equation $\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + (k_n(\omega))^2 \tilde{E}(z,\omega) = 0$

Dispersion parameter	Definition	Calculation from $n(\lambda)$
Phase Velocity v_p	$\frac{\omega}{k_n}$	$c_n = \frac{c}{n}$
Group velocity v_g	$\frac{d\omega}{dk_n} = \frac{c}{n_g}$	$\frac{c}{n}\frac{1}{1 - \frac{n'}{n}\lambda}$
Group delay T_g	$\frac{d\varphi}{d\omega} = L_d\frac{dk_n}{d\omega} = \frac{L_d}{v_g}$	$\frac{nL_d}{c}\left(1 - \frac{n'}{n}\lambda\right)$
Group delay dispersion $\frac{dT_g}{d\omega}$	$\frac{d^2\varphi}{d\omega^2}$	$\frac{\lambda^3 L_d}{2\pi c^2}n''$ [fs^2]
$\frac{dT_g}{d\lambda}$	$d\lambda = -\frac{\lambda^2}{2\pi c}d\omega$	$-\frac{\lambda L_d}{c}n''$ [fs/nm]
Third order dispersion	$\frac{d^3\varphi}{d\omega^3}$	$-\frac{\lambda^4 L_d}{4\pi^2 c^3}(3n'' + \lambda n''')$
Group refractive index n_g	$n_g = n + \omega\frac{dn}{d\omega}$	

4 Quantum Mechanics

Planck's constant $h = 6.626 \times 10^{-34}\,\mathrm{J\,s} = 4.136 \times 10^{-15}\,\langle\mathrm{s}\rangle, \hbar = \frac{h}{2\pi}$, Boltzmann's constant $k_B = 1.380\,649 \times 10^{-23}\,\mathrm{JK}^{-1}$, conversions: $1\,\mathrm{J} = 6.242 \times 10^{18}\,\mathrm{eV}$, $1\,\mathrm{eV} = 1.6022 \times 10^{-19}\,\mathrm{J}$

4.1 Blackbody Radiation

Planck's assumption $E = h\nu$

Planck's radiation law $E(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}, \quad E(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_BT}-1}$

Stefan-Boltzmann's law $E = \alpha T^4, \quad \alpha = 51.9504\pi k_B^4/c^3h^3 = 7.5643 \times 10^{-16}\,\mathrm{Jm^{-3}K^{-4}}$

Wien's displacement law $\lambda T = b, \quad b = hc/4.9651k_B = 2.8978 \times 10^{-3}\,\mathrm{m\,K}$

4.2 Photons and Electromagnetic Waves

Photon energy $E = h\nu = h\frac{c}{\lambda} = \hbar\omega = cp$

Photon momentum $p = \frac{h\nu}{c} = \frac{h}{\lambda} = \hbar k, \quad$ with the wavenumber $k = 2\pi/\lambda$

Photoelectric emission Let φ be the energy required for an electron to escape metal:

$$E_{\mathrm{kin}} = E_{\mathrm{tot}} - \varphi = h\nu - \varphi$$

Rayleigh scattering Elastic scattering of light: Causes only small perturbations to the atom, treat it like a classical oscillator excited by a wave. The outgoing photons have the same energy as the incoming photons.

Compton effect $\Delta\lambda = \lambda' - \lambda = \lambda_C(1-\cos\theta), \quad$ where for electrons $\lambda_C = \frac{h}{m_ec} = 2.4262\times10^{-12}\,\mathrm{m}$

Pondermotoric emission Kinetic energy of oscillating part of a free electron in a linearly polarized electromagnetic wave averaged over time:

$$U_p = \overline{E_{\mathrm{kin,osc}}} = \frac{1}{4}\frac{e^2}{m\omega^2}E_0^2, \quad U_p = 9.34\cdot10^{-8}\lambda^2 I$$

4.3 Particles and Matter Waves

Particle relativistic energy $E = c\sqrt{m_0^2c^2 + p^2}$

Particle relativistic momentum $p = \frac{m_0v}{\sqrt{1-v^2/c^2}} \quad \Leftrightarrow v = \frac{pc}{\sqrt{p^2+m_0^2c^2}}$

de Broglie wavelength (for matter waves) $\lambda = h/p, \quad p = \hbar k, \quad k = \frac{2\pi}{\lambda}$

Wavepacket $\omega(k) = \frac{p^2}{2m\hbar} = \frac{\hbar k^2}{2m}, \quad \psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} Ae^{i(kx-\omega(k)t)}dk$

Group velocity $v_0 = \frac{\hbar k_0}{m} = \frac{p_0}{m}$

Probability distribution $P(x,t)\mathrm{d}x = |\psi(x,t)|^2\mathrm{d}x, \quad |\psi(x,t)|^2 = \int_{-\infty}^{+\infty}\psi^*(x)\psi(x)\mathrm{d}x$
Probability of finding particle in $\mathrm{d}V = [r, r+\mathrm{d}r]$: $P(r)\mathrm{d}r = r^2|R(r)|^2\mathrm{d}r$

Normalization $\int_{-\infty}^{+\infty}|\psi(x,t)|^2\mathrm{d}x = 1$

Heisenberg Uncertainty Principle $\Delta x\Delta p_x \geq \frac{\hbar}{2}, \quad \Delta t\Delta E \gtrsim h$

4.4 Schrödinger Equation and Operators

	1-dimensional	general
Schrödinger Equation	$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$	$\hat{H}\boldsymbol{\Psi}(\boldsymbol{r},t) = i\hbar\frac{\partial}{\partial t}\boldsymbol{\Psi}(\boldsymbol{r},t)$
Time-independent SE	$\hat{H}\psi(x) = E\psi(x)$	$\hat{H}\boldsymbol{\Psi}(\boldsymbol{r}) = E\boldsymbol{\Psi}(\boldsymbol{r})$
Position operator	$\hat{x} = x$	$\hat{\boldsymbol{r}} = \boldsymbol{r}$
Momentum operator	$\hat{p} = -i\hbar\frac{\partial}{\partial x}$	$\hat{\boldsymbol{p}} = -i\hbar\nabla$
Kinetic Energy op.	$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$	$\hat{T} = \frac{\hat{\boldsymbol{p}}^2}{2m} = -\frac{\hbar^2}{2m}\triangle$
Potential Energy op.	$\hat{V} = V(x)$	$\hat{V} = V(\boldsymbol{r})$
Total energy op.	$\hat{E} = i\hbar\frac{\partial}{\partial t}$	$\hat{E} = i\hbar\frac{\partial}{\partial t}$
Hamiltonian	$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$	$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\boldsymbol{r})$

Particle in potential 1D $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$

Expected value for observable $\hat{O}$: $\langle\hat{O}\rangle = \int_V \psi^*(x,t)\hat{O}\psi(x,t)dV$
Probability of measuring the (eigen-)value c_n of a system $\psi = \sum c_n\psi_n$ is $|c_n|^2$

Commutator $\left[\hat{Q},\hat{O}\right] = \hat{Q}\hat{O} - \hat{O}\hat{Q}$

Two observable operators can share the same eigenstates if and only if $\left[\hat{Q},\hat{O}\right] = 0$.
For position and momentum $[\hat{x},\hat{p}] = i\hbar \quad \longrightarrow$ eigenstates are not shared.

Measurement probability of $\hat{O}$ yielding a value of O_m for wavefunction $\psi(x) = \sum_n a_n\psi_n(x)$:
$P(O_m) = \sum_j |a_j|^2$, where $j \in \{j \mid O_j = O_m\} \quad \longrightarrow \quad$ collapse to $\psi'(x) = \frac{\sum_j a_j\psi_j(x)}{\sqrt{\sum_j |a_j|^2}}$

Superposition of two time-dependent eigenstates $\psi(x,t) = c_1\psi_1(x)e^{-i\frac{E_1}{\hbar}t} + c_2\psi_2(x)e^{-i\frac{E_2}{\hbar}t}$
$|\psi(x,t)|^2 = |c_1\psi_1(x)|^2 + |c_2\psi_2(x)|^2 + 2c_1\psi_1c_2\psi_2\cos\left(\frac{E_2-E_1}{\hbar}t\right) \quad \longrightarrow \omega = \frac{E_2-E_1}{\hbar}$

4.5 Infinite Potential Well

Potential $V(x) = \begin{cases} 0, & 0 \leq x \leq a \\ \infty & \text{else} \end{cases}$

Energy eigenvalues $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{h^2}{8ma^2}n^2 \quad \longrightarrow$ Quantized wavelengths: $\lambda_n = \frac{2a}{n}$

Energy eigenstates $\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & 0 \leq x \leq a \\ 0 & \text{else} \end{cases}$

Dynamic solution $\Psi_n(x,t) = \psi_n(x)e^{-i\frac{E_n}{\hbar}t}$

4.6 Potential barrier (Tunneling)

Potential $V(x)) \begin{cases} V_0 & 0 \le x \le a \\ 0 & \text{else} \end{cases}$

Energy eigenstates $\psi_E(x) = \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} & x < 0 \\ A_2 e^{ik'x} + B_2 e^{-ik'x} & 0 \le x \le a \\ A_3 e^{ikx} + B_3 e^{-ikx} & x > a \end{cases}$ (determine A_i, B_i with requirement for continuity)

where $k = \sqrt{\frac{2mE}{\hbar^2}}$, $k' = \sqrt{k^2 - k_0^2}$, $k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$

Transmission and reflection $\frac{A_i}{A_t} = \frac{4kk'e^{-a(k-k')}}{(k+k')^2 - e^{2iak'}(k-k')^2}$, $\frac{A_i}{A_r} = \frac{(k^2-k'^2)\sin(ak')}{2ikk'\cos(ak') + (k^2+k'^2)\sin(ak')}$

Tunneling Probability Plane wave with $E < V_0$: $T = \frac{4E(E_0-E)}{4E(E_0-E)+E_0^2\sinh^2(\kappa a)}$, $\kappa = \sqrt{\frac{2m(E_0-E)}{\hbar^2}}$

4.7 Quantum Harmonic Oscillator

Harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2 \longrightarrow \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

Energy levels Eigenvalues of $\hat{H}\psi(x) = E_n\psi(x)$: $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

Energy eigenstates with the Hermite polynomials $H_n(x)$

$$\psi_n(x) = \left(\frac{m\omega}{\pi 2^{2n}(n!)^2\hbar}\right)^{1/4} e^{-m\omega^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

Dynamic solution $\Psi_n(x,t) = \psi_n(x)e^{-i\frac{E_n}{\hbar}t} = \psi_n(x)e^{-i(n+1/2)\omega t}$

Ladder operators $\hat{a}^\dagger = -i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}\hat{p} + \left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{x}$, $\hat{a} = i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}\hat{p} + \left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{x}$,
$\left[\hat{a}, \hat{a}^\dagger\right] = 1$, $\hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$, $\hat{a}\psi_n = \sqrt{n}\psi_n$

4.8 Angular Momentum and Spin

Orbital angular momentum operator $\hat{\boldsymbol{L}} = \left(\hat{L}_x, \hat{L}_y, \hat{L}_z\right) = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} = \boldsymbol{r} \times -i\hbar\nabla$

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad \hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}\right) \quad \hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\hat{\boldsymbol{L}}^{\boldsymbol{2}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2 \triangle_{S^2}$$

Spin angular momentum operator $\hat{\boldsymbol{S}} = \left(\hat{S}_x, \hat{S}_y, \hat{S}_z\right) = \frac{\hbar}{2}\boldsymbol{\sigma}$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y \quad \hat{S}_z = \frac{\hbar}{2}\sigma_z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Total angular momentum operator $\hat{\boldsymbol{J}} = \left(\hat{J}_x, \hat{J}_y, \hat{J}_z\right) = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}} = -i\hbar\boldsymbol{r} \times \nabla + \frac{\hbar}{2}\boldsymbol{\sigma}$

$$\hat{J}_x = \hat{L}_x + \hat{S}_x \quad \hat{J}_y = \hat{L}_y + \hat{S}_y \quad \hat{J}_z = \hat{L}_z + \hat{S}_z$$

Commutations $\left[\hat{L}_x, \hat{L}_y\right] = i\hbar L_z$, $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar L_x$, $\left[\hat{L}_z, \hat{L}_x\right] = i\hbar L_y$
$\left[\hat{\boldsymbol{L}}^{\boldsymbol{2}}, \hat{L}_x\right] = \left[\hat{\boldsymbol{L}}^{\boldsymbol{2}}, \hat{L}_y\right] = \left[\hat{\boldsymbol{L}}^{\boldsymbol{2}}, \hat{L}_z\right] = 0$

Same goes for spin and total angular momentum: $\left[\hat{S}_z, \hat{S}_x\right] = i\hbar S_y$, $\left[\hat{J}_z, \hat{J}_x\right] = i\hbar J_y$ $(\dots)$
$\left[\hat{\boldsymbol{S}}^{\boldsymbol{2}}, \hat{S}_i\right] = \left[\hat{\boldsymbol{J}}^{\boldsymbol{2}}, \hat{J}_i\right] = 0$

Uncertainty We can only determine L_z and $|\hat{\boldsymbol{L}}|^2$ $\longrightarrow \Delta L_x \cdot \Delta L_y \ge \frac{1}{2}\hbar L_z$

Angular momentum eigenstates

$\hat{\boldsymbol{L}}^{\boldsymbol{2}}Y_{lm_l}(\theta,\varphi) = L^2Y_{lm_l}(\theta,\varphi)$ $\hat{L}_zY_{lm_l}(\theta,\varphi) = L_zY_{lm_l}(\theta,\varphi)$ $\hat{\boldsymbol{S}}^{\boldsymbol{2}}\chi_{m_s} = S^2\chi_{m_s}$ $\hat{S}_z\chi_{m_s} = S_z\chi_{m_s}$

L^2	L_z	S^2	S_z	J^2	J_z
$\hbar^2 l(l+1)$	$\hbar m_l$	$\hbar^2 s(s+1)$	$\hbar m_s$	$\hbar^2 j(j+1)$	$\hbar m_j$
$l = 0, 1, \dots$	$m_l = -l, \dots, l$	$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$	$m_s = -s, \dots, s$	$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$	$m_j = -j, \dots, j$

Adding angular momenta $j = j_1 + j_2, j_1 + j_2 - 1, \dots |j_1 - j_2|$
for electrons $j = l \pm \frac{1}{2} > 0$

Fermions Particles with half-integer spin $s = \frac{1}{2}$: Electron, Proton, Neutron

Bosons Particles with integer spin $s = 1$: Photon

Magnetic moment operator for electrons: Bohr Magneton $\mu_B = \frac{e\hbar}{2m_e}$ and $g_s \approx 2.00232$
From spin: $\hat{\boldsymbol{\mu}}_{\boldsymbol{S}} = -\frac{g_s\mu_B}{\hbar}\hat{\boldsymbol{S}}$ $\mu_{S_z} = -\mu_B m_s$
From orbital angular momentum: $\hat{\boldsymbol{\mu}}_{\boldsymbol{L}} = -\frac{e}{2m_e}\hat{\boldsymbol{L}} = -\frac{\mu_B}{\hbar}\hat{\boldsymbol{L}}$ $\mu_{L_z} = -\frac{e}{2m_e}L_z = -\mu_B m_l$

Spin magnetic moment of Protons and Neutrons $\mu_K = \frac{e\hbar}{2m_p}$, $g_p \approx -5.58$, $g_n \approx 3.826$
From spin: $\hat{\boldsymbol{\mu}}_{\boldsymbol{p}} = -\frac{g_p\mu_K}{\hbar}\hat{\boldsymbol{S}}$ $\hat{\boldsymbol{\mu}}_{\boldsymbol{n}} = -\frac{g_n\mu_K}{\hbar}\hat{\boldsymbol{S}}$

Stern-Gerlach experiment Showed that spin is either parallel or antiparallel to B-field.
$\boldsymbol{F} = -\nabla(\boldsymbol{\mu}_s \cdot \boldsymbol{B})$, $V = \boldsymbol{\mu}_s \cdot \boldsymbol{B}$

5 Hydrogen Atom

For hydrogen atom $Z = 1$, Bohr radius $a_0 = 5.29 \times 10^{-11}\,\text{m} = 0.529\,\text{Å}$, $1\,\text{Å} = 1 \times 10^{-10}\,\text{m}$

Coulomb potential $E_p(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$

Hamilton operator Nucleus at rest approximation $(m_p \gg m_e)$ $\hat{H} = \frac{\hat{p}^2}{2m} + E_p(r)$

S.E. Nucleus at rest $(m_p \gg m_e)$ $\left(-\frac{\hbar^2}{2m}\triangle - \frac{Ze^2}{4\pi\varepsilon_0 r}\right)\psi = E\psi$

S.E. Nucleus *not* at rest $\left(-\frac{\hbar^2}{2m_e}\triangle_e - \frac{\hbar^2}{2m_p}\triangle_p - \frac{Ze^2}{4\pi\varepsilon_0|\boldsymbol{r_e}-\boldsymbol{r_p}|}\right)\psi(\boldsymbol{r_e},\boldsymbol{r_p}) = E\psi(\boldsymbol{r_e},\boldsymbol{r_p})$

Eigenvalues $E_n = -13.606\,\text{eV} \cdot \frac{Z^2}{n^2}$, $n = 1, 2, 3\dots$

Eigenstates $\psi_{nlm_lm_s}(r,\theta,\varphi) = R_{nl}(r)Y_{lm_l}(\theta,\varphi)\chi_{m_s}$
- Principle quantum number $n = 1, 2, 3\dots$ («Hauptquantenzahl»)
- Orbital quantum number $l = 0, \dots, n-1$, n values («Drehimpulsquantenzahl»)
- Oribtal magnetic quantum number $m_l = -l, \dots, 0, \dots, l$, $2l+1$ values («Richtungsquantenzahl des Drehimpulses»)

- Spin quantum number $m_s = -s, \dots, s$, for electrons $s = \frac{1}{2} \to m_s = -\frac{1}{2}, +\frac{1}{2}$ («Spinquantenzahl»)

Time-dependent eigenstate $\Psi_{nlm_l}(r,\theta,\varphi,t) = R_{nl}(r)Y_{lm_l}(\theta,\varphi)e^{-i\frac{E_n}{\hbar}t}$

Centrifugal potential $E_{p,z}(r) = \frac{l(l+1)\hbar^2}{2mr^2}$

Radial Schrödinger equation $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}R_{nl}(r) + \left(-\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{l(l+1)\hbar^2}{2mr^2}\right)R_{nl}(r) = E_n R_{nl}(r)$

Radial eigenfunction $R_{nl}(r) = \sqrt{\left(\frac{2Z}{na_0}\right)^3 \frac{(n-l-1)!}{2n(n+1)!}}e^{-\rho/2}\rho^l L_{2l+1}^{n-l-1}(\rho)$

where $\rho = \frac{2Zr}{na_0}$, $L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^{n+\alpha})$

Radial probability distribution $P(r)\mathrm{d}r = r^2|R(r)|^2\mathrm{d}r$

Average radius $\langle r\rangle = \int \psi^* r\psi \mathrm{d}V = \int r|\psi|^2\mathrm{d}V$, $\langle r\rangle|_{nl} = n^2\left(1+\frac{1}{2}\left(1-\frac{l(l+1)}{n^2}\right)\right)\frac{a_0}{Z}$

$\langle r\rangle|_d < \langle r\rangle|_p < \langle r\rangle|_s$

5.1 The Bohr Model

Bohr radius $r_n = \frac{n^2}{Z}a_0 = 2\pi\varepsilon_0\frac{\hbar^2}{Zm_e e^2}n^2$, $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}$

Rydberg energy $E_{Ry} = \frac{m_e e^4}{32\pi^2\varepsilon_0^2\hbar^2} = 2.180\times10^{-18}\,\mathrm{J} = 13.606\,\mathrm{eV}$

Energy levels $E_n = -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{a_0}\cdot\frac{Z^2}{n^2} \approx E_{Ry}\cdot\frac{Z^2}{n^2}$, $n = 1,2,3\dots$

Quantized angular momentum $L_n = m_l v r_n = n\hbar$

6 Atoms with more than one Electron

Hamilton operator $\hat{H} = \sum_{i=1}^N\left(\frac{\hat{\boldsymbol{p}}_i^2}{2m_e} - \frac{Ze^2}{4\pi\varepsilon_0 r_i}\right) + \sum_{i\neq j}\frac{e^2}{4\pi\varepsilon_0 r_{ij}}$

6.1 Exchange Degeneracy

Exchange operator $\hat{X}\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \psi(\boldsymbol{r}_2,\boldsymbol{r}_1)$, $\hat{X}\psi_X = X\psi_X$, $X = \pm1$

- For **fermions (half-integer spin particles)**, the wavefunction must be *antisymmetric with respect to exchange of particle labels* ($X=-1$).
- For **bosons (integer spin particles)**, the wavefunction must be *symmetric with respect to exchange of particle labels* ($X=1$).

Pauli exclusion principle The requirement that two identical fermions must be in anti-symmetric wavefunctions means that they *cannot be in the same state.*

Hund's rule In their ground state, atoms have maximum total spin, which is allowed by the pauli exclusion principle

6.2 Helium Atom (Z=2)

Potential assuming ($m_k \gg m_e$) $E_p = \underbrace{-\frac{Ze^2}{4\pi\varepsilon_0 r_1} - \frac{Ze^2}{4\pi\varepsilon_0 r_2}}_{\text{nucleus}-e^-\text{attraction}} + \underbrace{\frac{e^2}{4\pi\varepsilon_0 r_{12}}}_{e^--e^-\text{attraction}}$

Hamiltonian $\hat{H} = \sum_{i=1}^2\frac{\hat{\boldsymbol{p}}_i^2}{2m} - \sum_{i=1}^2\frac{2e^2}{4\pi\varepsilon_0 r_i} + \frac{e^2}{4\pi\varepsilon_0 r_{12}}$

Without electron-electron interaction: $\hat{H} = \frac{\hat{\boldsymbol{p}}_1^2}{2m} - \frac{2e^2}{4\pi\varepsilon_0 r_1} + \frac{\hat{\boldsymbol{p}}_2^2}{2m} - \frac{2e^2}{4\pi\varepsilon_0 r_2}$

$\longrightarrow$ Each electron is described by hydrogenic wave function

Effective nuclear charge $Z_{\text{eff}} = Z - \delta$, $\delta_{\text{He}} = 0.32$, $Z_{\text{eff,He}} = 1.6875$

Bonding energy $E = 2(Z-\delta)^2 E_H$, $E_{\text{Tot,He}} = -2(Z_{\text{eff}})^2\cdot 13.6\,\mathrm{eV}$

Ground state $\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_2)\chi_-$, $\chi_\pm = \frac{1}{\sqrt{2}}(\chi_{\uparrow_1\downarrow_2} \pm \chi_{\downarrow_1\uparrow_2})$

Both electrons in lowest energy spatial eigenfunction, opposite spin states, anti-symmetric wavefunction, anti-symmetric spin wavefunction.

First excited state

Singlet: $\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}(\psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_2) + \psi_1(\boldsymbol{r}_2)\psi_2(\boldsymbol{r}_1))\chi_-$

Triplets: $\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}(\psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_2) - \psi_1(\boldsymbol{r}_2)\psi_2(\boldsymbol{r}_1))\chi_+$

$\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}(\psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_2) - \psi_1(\boldsymbol{r}_2)\psi_2(\boldsymbol{r}_1))\chi_{\uparrow_1\uparrow_2}$

$\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}(\psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_2) - \psi_1(\boldsymbol{r}_2)\psi_2(\boldsymbol{r}_1))\chi_{\downarrow_1\downarrow_2}$

6.3 Pertubation theory

$\hat{H} = \hat{H}_0 + \delta\hat{H}$, $\delta\hat{H} \ll \hat{H}_0$ $\longrightarrow$ $\hat{H}\psi_n = (E_n+\delta E)\psi_n$, $\delta E = \langle\delta\hat{H}\rangle$, $\delta E_n = \int\psi_n^*\delta\hat{H}\psi_n\mathrm{d}x$

Commutation requirement $\left[\hat{H}_0, \hat{H}_0+\delta\hat{H}\right] = 0$

Fine-structure constant $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}$

Spin-orbit coupling $\delta V = -\boldsymbol{\mu}_s\cdot\boldsymbol{B}_{\text{nucleus}}$, $\boldsymbol{B}_{\text{nucleus}} = \frac{1}{2}\frac{\mu_0}{4\pi}\frac{Ze}{r^3 m_e}\boldsymbol{L}$

$\delta\hat{H} = \frac{g_s\mu_0 Ze^2}{16\pi m_e^2 r^3}\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{L}}$, $\delta E_{nljs} = \frac{1}{4}\alpha^4 m_e c^2 Z^4\frac{j(j+1)-l(l+1)-s(s+1)}{n^3 l(l+1/2)(1+1)}$

Spin-orbit operator $2\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{L}} = \hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2$, $\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{L}}\psi_{jls} = \frac{\hbar^2}{2}\left(j(j+1) - l(l(+1) - s(s+1)\right)\psi_{jls}$

Magnetic torque & Potential $\boldsymbol{M} = \boldsymbol{\mu}\times\boldsymbol{B}$, $E_{\text{pot}} = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\mu B\cos\theta$, $\hat{E}_{\text{pot}} = -\hat{\mu}_z B$

Zeeman effect Interaction of orbital magnetic moment with weak external B-field.

$\delta\hat{H} = -\hat{\mu}_{L_z}B$, $E_{nlm_l} = E_n + \delta E = E_n - \mu_B m_l B$, $\Delta\nu_{tr} = \frac{\mu_B B}{h}$

Paschen-Back-Effekt Energy of electron in strong B-filed (such that L-S coupling is negligible)

$\Delta E_B = \mu_B B(m_l + 2m_s)$

Relativistic correction $\delta E = -\frac{1}{8}Z^2\alpha^4 m_e c^c\left(\frac{4n}{l+1/2} - 3\right)$

Optical transitions electric dipole approximation: $\hat{H}_1(t) = e\hat{\boldsymbol{r}}\cdot\boldsymbol{E}(t) = E_0 e^{-\omega t}(\hat{x}\pm i\hat{y}) + c.c.$

Selection rules: $l' = l\pm1$, right/left circular: $m' = m\pm1$, along z: $m' = m$

7 Molecules

Born-Oppenheimer approximation As the nucleus is much more massive than the electrons, the nucleus together with the core electrons can be treated separately from the valence electrons in the outermost shell. $\longrightarrow$ Solve Schrödinger equation for each electron with stationary nucleus.

Homonuclear diatomic molecules Molecule made of two atoms of the same element (H_2, N_2, etc.). Have inversion symmetry about center of molecule, with $\psi = \psi_1 \pm \psi_2$ either symmetric (+) or anti-symmetric (−).

7.1 Hydrogen Ion H_2^+

Potential $E_p = \frac{e^2}{4\pi\varepsilon_0}\left(-\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r}\right)$, r_1: $p_1 \leftrightarrow e^-$, r_2: $p_2 \leftrightarrow e^-$, r: $p_1 \leftrightarrow p_2$

Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\triangle + \frac{e^2}{4\pi\varepsilon_0}\left(-\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r}\right)$

Energy of Electron $E = E_a + \frac{e^2}{4\pi\varepsilon_0 r} - \frac{A\pm B}{1\pm S}$, + for ψ symmetric, − for ψ anti-symmetric
Overlapintergral: $S(r) = \int \psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_1)\mathrm{d}V_1$
Coulombintegral: $A(r) = \frac{e^2}{4\pi\varepsilon_0}\int \frac{\psi_1^2(\boldsymbol{r}_1)}{r_2}\mathrm{d}V_1$ (attraction between the two electrons)
Overlapping term: $B(r) = \frac{e^2}{4\pi\varepsilon_0}\int \frac{\psi_1(\boldsymbol{r}_1)\psi_2(\boldsymbol{r}_2)}{r_1}\mathrm{d}V_1$

Covalent bond Stable bond with minimal binding energy is achieved with a symmetric wavefunction $\psi = \psi_1 + \psi_2$

7.2 Hydrogen Molecule H_2

Potential $E_p = \frac{e^2}{4\pi\varepsilon_0}\left(-\frac{1}{r_1} - \frac{1}{r_1'} - \frac{1}{r_2} - \frac{1}{r_2'} + \frac{1}{r_{12}} + \frac{1}{r}\right)$

Hydrogen bonds Symmetric (σ_g) and anti-symmetric (σ_u) wavefuntions. * means non-bonding.
ns-orbitals, $|m_l| = 0 \rightarrow \sigma_g(ns)$, $\sigma_u^*(ns)$, np_z-orbitals, $|m_l| = 0 \rightarrow \sigma_g(np_z)$, $\sigma_u^*(np_z)$,
np_{xy}-orbitals, $|m_l| = 1 \rightarrow \pi_g^*(np_{xy})$, $\pi_u(np_{xy})$
$\longrightarrow$ Pauli exclusion principle with fermions: Total wavefunction must be anti-symmetric, select spin wavefunction appropriately $\psi_{\text{tot}} = \psi_{g,u} \cdot \chi_{u,g}$

Angular momentum states Except for σ-states, all states are twice degenerate (±, spin)

m_l:	0	±1	±2	±3	…
λ:	0	1	2	3	…
Symbol:	σ	π	δ	φ	…

7.3 Polyatomic Molecules

Bonds get formed, where the wavefunctions have the most overlap.

sp Hybrid orbital can increase wavefunction overlap and form stronger bonds.

Heteronucelar diatomic molecules Molecule made of two different atoms. Uneven electron distribution creates polar bond.

Polar bond Electron pairs are closer to one atom, which creates a partial negative charge δ^-.
$\psi = c_A\psi_A + c_B\psi_B$

Ionic bond Electrons move from one atom to another, creating a positive and a negative ion, which get held together by the coulomb force. For A^+B^-, $c_A = 0$, $c_B = 1$

Electron affinity Energy required to unbind electron from negatively charged ion $X^- = X + e^-$. Atoms with strong electron affinities tend to form ionic bonds, as this results in a lower energy state.

7.4 Molecular Rotation & Vibration

Let $B = \frac{\hbar}{4\pi Ic}$ $[\mathrm{cm}^{-1}]$

Rotational energy states $E_r = \frac{L^2}{2I} = \frac{\hbar^2}{2I}l(l+1) = Bhcl(l+1)$ $I = \mu r_0^2$, $\mu = \frac{m_1 m_2}{m_1+m_2}$

Energy difference between rotational states $\Delta E_{r,l+1,l} = \frac{\hbar^2}{I}(l+1) = 2Bhc(l+1)$

Allowed rotational transitions $\Delta l = \pm 1$

Emitted/absorbed radiation from rotational transition $\nu = \frac{\Delta E}{h} = 2Bc(l+1)$

Vibrational energy states $E_v = (v + \frac{1}{2})\hbar\omega_0$, $v = 0, 1, 2, 3, \ldots$
Ground state energy: $E_0 = \frac{1}{2}\hbar\omega_0$

Allowed vibrational transitions $\Delta v = \pm 1$

Total molecular energy $E = E_e + E_v + E_r = E_e + (v + \frac{1}{2})\hbar\omega_0 + \frac{\hbar^2}{2I}l(l+1)$

Energy change for electrical transition $\Delta E = E'' - E' = \Delta E_e + \Delta E_v + \Delta E_r$
- Electrical energy change: $\Delta E_e = E_e'' - E_e'$
- Vibrational energy change: $\Delta E_v = E_v'' - E_v' = (v'' + \frac{1}{2})\hbar\omega_0'' - (v' + \frac{1}{2})\hbar\omega_0'$
- Rotational energy change: $\Delta E_r = E_r'' - E_r' = \frac{\hbar^2}{2I''}l''(l''+1) - \frac{\hbar^2}{2I'}l'(l'+1)$

Emitted/absorbed radiation from electrical transition
$\nu = \frac{\Delta E}{h} = \nu_e + \nu_v(v'', v') + \nu_r(l'', l')$, $\nu_e = \frac{\Delta E_e}{h}$

Selection rules for electrical transitions
- Rotational: $\Delta l = 0, \pm 1$, except for $l'' = 0 \leftrightarrow l' = 0$
- Spin: $\Delta S = 0$
- Vibrational: No rule exists. Transitions are most likely to occur at the reversal point of the vibrating atoms, where they are the same for the initial and end state. This way the vibrational wavefunctions get distorted the least during the transition.

8 Statistical Physics

Boltzmann-constant $k = 1.3805 \times 10^{23}\,\mathrm{J\,K^{-1}} = 8.6178 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$, Avocadro's number $N_A = 6.0225 \times 10^{23}\,\mathrm{mol^{-1}}$

	Maxwell-Boltzmann	Fermi-Dirac	Bose-Einstein
Particle	Classical	Fermions	Bosons
Pauli-Principle	×	✓	×
Identical	✓	✓	✓
Distinguishable	✓	×	×

8.1 Maxwell-Boltzmann Distribution

n_i: number of particles in energy state E_i. g_i: probability of finding particle in energy state E_i.
The probability of a certain distribution is proportional to number of possibilities to distribute the particles into the available energy states.

Distribution probability distinguishable: $P = N! \prod_{i=1}^{N} \frac{g_i^{n_i}}{n_i!}$ indistinguishable: $P = \prod_{i=1}^{N} \frac{g_i^{n_i}}{n_i!}$

Conservation of particle number and Energy $\sum_i \mathrm{d}n_i = 0, \quad \sum_i E_i \mathrm{d}n_i = 0$

Mean of physical property $F(E_i) \quad F_{\text{ave}} = \frac{1}{N}\left(\sum_i n_i F(E_i)\right)$

Most probable distribution (equilibrium) $n_i = g_i e^{-\alpha-\beta E_i} = \frac{N}{Z} g_i e^{-\beta E_i}$

System in statistical equilibrium

Total number of particles $N = e^{-\alpha} Z$

Total energy $U = \frac{N}{Z}\left(\sum_i g_i E_i e^{-\beta E_i}\right) = -N \frac{\mathrm{d}}{\mathrm{d}\beta}(\ln Z) = kNT^2 \frac{\mathrm{d}}{\mathrm{d}T} \ln Z$

Average Energy of a particle $E_{\text{ave}} = \frac{U}{N} - \frac{\mathrm{d}}{\mathrm{d}\beta}(\ln Z) = kT^2 \frac{\mathrm{d}}{\mathrm{d}T} \ln Z$

Absolute Temperature $kT = \frac{1}{\beta}$

Equilibrium partition function $Z = \sum_i g_i e^{-\beta E_i} Z = \sum_i g_i e^{-E_i/kT}$

Maxwell-Boltzmann distribution $n_i = \frac{N}{Z} g_i e^{-E_i/kT}$ (equilibrium distribution)

Mean of physical property $F(E_i) \quad F_{\text{ave}} = \frac{1}{Z}\sum_i g_i F(E_i) e^{-\beta E_i} = \frac{1}{Z}\sum_i g_i F(E_i) e^{-E_i/kT}$

Application: Ideal Gas

Partition integral $Z = \int_0^\infty e^{-E/kt} g(E)\mathrm{d}E = \frac{V(2\pi m kT)^{3/2}}{h^3}$ (for continuous energy states)

Number of molecule states between E and $E + \mathrm{d}E$ $g(E)\mathrm{d}E = \frac{4\pi V (2m^3)^{1/2}}{h^3} E^{1/2} \mathrm{d}E$

Mean energy of molecule $E_{\text{ave}} = \frac{3}{2}kT$

Total energy $U = N E_{\text{ave}} = \frac{3}{2}kNT = \frac{3}{2}\mathcal{N}kN_A T = \frac{3}{2}\mathcal{N}RT, \quad \mathcal{N} = \frac{N}{N_A}, \quad R = kN_A$

Energy distribution for molecules $\mathrm{d}n = \frac{N}{Z} e^{-E/kT} g(E)\mathrm{d}E, \quad \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{2\pi N}{(\pi kT)^{3/2}} E^{1/2} e^{-E/kT}$

Velocity distribution for molecules $\frac{\mathrm{d}n}{\mathrm{d}v} = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$

Average velocity $\langle v \rangle = \sqrt{\frac{8kT}{\pi m}}$

8.2 Fermi-Dirac Distribution

g_i is the maximal number of particles (fermions) in energy state E_i allowed by the exclusion principle

Distribution probability $P = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}$

Fermi-Dirac distribution law $n_i = \frac{g_i}{e^{\alpha+\beta E_I}+1} = \frac{g_i}{e^{(E_i-\varepsilon_F)/kT}+1}$ (most probable distribution)
For $T = 0$, all energy states up to $E = \varepsilon_F$ fully occupied ($n_i = g_i$), for $E > \varepsilon_F$ empty ($n_i = 0$).

Fermion temperature $kT = \frac{1}{\beta}$

Total chemical potential $\varepsilon_F = -\alpha kT$

Fermi temperature $\Theta_F = \frac{\varepsilon_F}{k}$

Application: Electron Gas

Number of electrons between E and $E + \mathrm{d}E$ $\mathrm{d}n = \frac{g(E)\mathrm{d}E}{e^{(E-\varepsilon_F)/kT}+1}$

Number of electron states between E and $E + \mathrm{d}E$ $g(E)\mathrm{d}E = \frac{8\pi V (2m^3)^{1/2}}{h^3} E^{1/2} \mathrm{d}E$

Fermi-Dirac formula for energy distribution $\frac{\mathrm{d}n}{\mathrm{d}E} = \frac{8\pi V (2m^3)^{1/2}}{h^3} \frac{E^{1/2}}{e^{(E-\varepsilon_F)/kT}+1}$

Fermi energy $\varepsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$

8.3 Bose-Einstein Distribution

Distribution probability $P = \prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$

Bose-Einstein distribution law $n_i = \frac{g_i}{e^{\alpha+\beta E_i}-1} = \frac{g_i}{e^{\alpha+E_i/kT}-1}$

Boson temperature $kT = 1/\beta$

Application: Photon Gas

Number of photons between E and $E + \mathrm{d}E$ $\mathrm{d}n = \frac{g(E)\mathrm{d}E}{e^{E/kT}-1} = \frac{8\pi V}{c^3} \frac{\nu^2 \mathrm{d}\nu}{e^{h\nu/kT}-1}$

Number of photon states in interval $\mathrm{d}\nu/\mathrm{d}E$ $g(E)dE = g(\nu)\mathrm{d}\nu = \frac{8\pi V}{c^3}\nu^2 \mathrm{d}\nu$

Energy density distribution of black body $E(\nu) = \frac{h\nu}{V}\frac{\mathrm{d}n}{\mathrm{d}v} = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}$ (Planck)

9 Analysis

9.1 Vector Analysis

Satz von Gauß $\int_{\partial V} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{A} = \int_V \operatorname{div} \boldsymbol{F}\, \mathrm{d}V$

Satz von Stokes $\oint_{\partial A} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_A \operatorname{rot} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{A}$

Cartesian coordinates

Gradient $\boldsymbol{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

Laplace $\triangle = \boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Divergence $\operatorname{div} \boldsymbol{F} \coloneqq \boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z$

Curl $\operatorname{rot} \boldsymbol{F} \coloneqq \boldsymbol{\nabla} \times \boldsymbol{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix}$

Cylindrical coordinates

$$x = \rho k\cos(\varphi), \quad y = \rho\sin(\varphi), \quad z = z, \quad r \geq 0, \varphi \in \mathbb{R}/2\pi\mathbb{Z}$$

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cdot\cos(\varphi) \\ \rho\cdot\sin(\varphi) \\ z \end{pmatrix} \qquad \mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z$$

Divergence $\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}$

Curl $\boldsymbol{\nabla}\times\boldsymbol{V} = \left[\frac{1}{r}\frac{\partial V_z}{\partial\varphi} - \frac{\partial V_\varphi}{\partial z}\right]\boldsymbol{e}_r + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\boldsymbol{e}_\varphi + \frac{1}{r}\left[\frac{\partial(rV_\varphi)}{\partial r} - \frac{\partial V_r}{\partial\varphi}\right]\boldsymbol{e}_z$$

Spherical coordinates

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \quad r \geq 0, 0 \leq \theta \leq \pi, \varphi \in \mathbb{R}/2\pi\mathbb{Z}$$

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2\mathrm{d}r\mathrm{d}\Omega(\theta,\varphi) = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

Gradient $\boldsymbol{\nabla} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \end{pmatrix}$

Laplace $\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{S^2}, \quad \triangle_{S^2} = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$

Divergence $\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{1}{r^2}\frac{\partial(r^2V_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(\sin(\theta)V_\theta)}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial V_\varphi}{\partial\varphi}$

Curl $\boldsymbol{\nabla}\times\boldsymbol{V} = \frac{1}{r\sin(\theta)}\left[\frac{\partial(\sin(\theta)V_\varphi)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right]\boldsymbol{e}_r + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial V_r}{\partial\varphi} - \frac{\partial(rV_\varphi)}{\partial r}\right]\boldsymbol{e}_\theta + \frac{1}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\boldsymbol{e}_\varphi$

9.2 Äquivalenzen

$$\sin(a)+\sin(b) = 2\cdot\sin\left(\tfrac{a+b}{2}\right)\cdot\cos\left(\tfrac{a-b}{2}\right)$$
$$\cos(a)+\cos(b) = 2\cdot\cos\left(\tfrac{a+b}{2}\right)\cdot\cos\left(\tfrac{a-b}{2}\right)$$

$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$$
$$\sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}$$
$$\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}$$

9.3 Differentialgleichungen

2. Ordnung, homogen mit konstanten Koeffizienten $ay'' + by' + cy = 0$

Die Lösungen der zugehörigen charakteristischen Gleichung $a\lambda^2 + b\lambda + c = 0$ seien $\lambda_{1,2}$.

- $\lambda_{1,2} \in \mathbb{R}$ und $\lambda_1 \neq \lambda_2$ $\quad y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
- $\lambda_1 = \lambda_2$ $\quad y(x) = (C_1 + C_2x)e^{\lambda_1 x}$
- $\lambda_{1,2} = \alpha \pm i\beta,\ \beta \neq 0$ $\quad y(x) = (C_1\sin\beta x + C_2\cos\beta x)e^{\alpha x}$

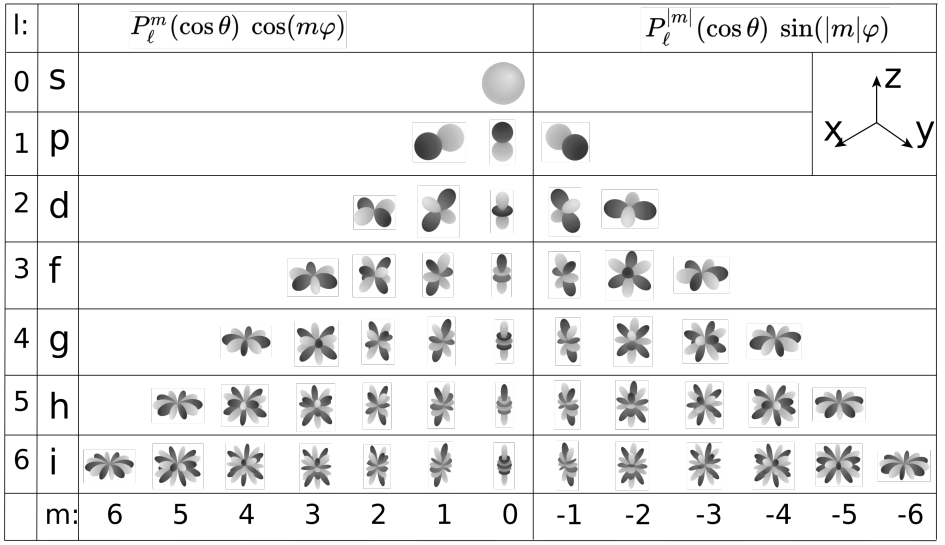
Ansätze $\alpha > 0$

$y'' + \alpha^2 y = 0 \quad \longrightarrow \quad y(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$

$y'' - \alpha^2 y = 0 \quad \longrightarrow \quad y(x) = Ae^{\alpha x} + Be^{-\alpha x}$

9.4 Kugelfunktionen

Plots show $|Y_{lm_l}(\theta,\varphi)|^2$



$$\left[\hat{A}, c\right] = 0$$
$$\left[\hat{A}, \hat{A}\right] = 0$$
$$\left[\hat{A}, \hat{B}\right] = -\left[\hat{B}, \hat{A}\right]$$
$$\left[\hat{A}, \textstyle\sum_n c_n\hat{A}^n\right] = 0$$

$$\left[c\hat{A}, \hat{B}\right] = \left[\hat{A}, c\hat{B}\right] = c\left[\hat{A}, \hat{B}\right]$$
$$\left[\hat{A}, \hat{B} \pm \hat{C}\right] = \left[\hat{A}, \hat{B}\right] \pm \left[\hat{A}, \hat{C}\right]$$
$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$$
$$\left[\hat{A}, \hat{B}\hat{C}\right] = \hat{B}\left[\hat{A}, \hat{C}\right] + \left[\hat{A}, \hat{B}\right]\hat{C}$$

9.5 Kommutator identitäten

9.6 Unit powers

Prefix	Shorthand	Power	Prefix	Shorthand	Power
Deka	da	10^{1}	Dezi	d	10^{-1}
Hekto	h	10^{2}	Centi	c	10^{-2}
Kilo	k	10^{3}	Milli	m	10^{-3}
Mega	M	10^{6}	Mikro	μ	10^{-6}
Giga	G	10^{9}	Nano	n	10^{-9}
Tera	T	10^{12}	Pico	p	10^{-12}
Peta	P	10^{15}	Femto	f	10^{-15}
Exa	E	10^{18}	Atto	a	10^{-18}
Zetta	Z	10^{21}	Zepto	z	10^{-21}
Yotta	Y	10^{24}	Yokto	y	10^{-24}