1 Groups

Finite Groups:

- Cyclic Group $\mathbb{Z}/n\mathbb{Z}, \quad |\mathbb{Z}_n| = n.$
- Symmetric Group $|S_n| = n!$
- Dihedral Gr $D_n = \{1, R, .., R^{n-1}, S, RS, .., R^{n-1}S\}, |D_n| = 2n.$

Lie Groups ($\mathbb{C}^\infty$-Manifolds):

- $\mathrm{GL}(n, K) = \{\text{invertible } n \times n \text{ matricies}\}$
- $\mathrm{GL}(V) = \{\text{invertible linear mappings } V \to V\}$
- $\mathrm{SL}(n) = \{A \in \mathrm{GL}(n, K) |\det(A) = 1\}$
- $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) | A^\top = A^{-1}$ $\iff (Ax, Ay) = (x, y)\ \forall x, y \in \mathbb{R}\}$, where $(x, y) = \sum_{j=1}^n x_j y_j$
- $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) | A^* = A^{-1}$ $\iff (Az, Aw) = (z, w)\ \forall z, w \in \mathbb{C}\}$, where $(z, w) = \sum_{j=1}^n \bar{z}_j w_j$
- $\mathrm{O}(p, q) = \{A \in \mathrm{GL}(p + q, \mathbb{R}) | (Ax, Ay)_{p,q} = (x, y)_{p,q}\}$, where $(x, y)_{p,q} = \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$
- $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) |\det(A) = 1\}$
- $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) |\det(A) = 1\}$
  $$A \in \mathrm{SU}(2) \ \Rightarrow\ A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},\ (\alpha, \beta) \in \mathbb{C}^2,\ |\alpha|^2 + |\beta|^2 = 1$$
  $$= \alpha\sigma_0 + i(\beta\sigma_1 + \gamma\sigma_2 + \delta\sigma_3),\ |\alpha|^2 + |\beta|^2 = 1$$
- $\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(n, \mathbb{R}) | w(Ax, Ay) = w(x, y)\}$, where $w(x, y) = \sum_{i=1}^n (X_{2i-1} Y_{2i} - X_{21} Y_{2i-1})$

Definitions:

- A **group** G with a product $G \times G \to G$, $(g, h) \mapsto gh$ has:
  - *associativity* $\quad a(bc) = (ab)c$
  - *neutral element* $\quad 1 \in G : 1g = g1 = g$
  - *inverse* $\quad \forall g \in G \exists g^{-1} \in G : gg^{-1} = 1 = g^{-1}g$
- A **homomorphism** $\varphi : G \to H$ fulfills $\varphi(g \circ_G g') = \varphi(g) \circ_H \varphi(g')\ \forall g, g' \in G$. It follows that $\varphi(1) = 1$, $\varphi(g^{-1}) = \varphi(g)^{-1}$. If φ is bijective, we call φ an **isomorphism** and write $G \cong H$.
- The set G/H of the **cosets** (**links-Nebenklassen**) of $H \subset G$ are the equivalence classes of $g_1 \sim g_2 \Leftrightarrow \exists h \in H$ so $g_2 = g_1 h$.
- A **normal subgroup/conjugacy class** of G is a subgroup H, so that $ghg^{-1} \in H\ \forall g \in G,\ h \in H$.

Useful Theorems:

- Let $H \subset G$ be a normal subgroup. Then G/H is a Group with product $[g_1][g_2] := [g_1 g_2]$.

2 Representations of Groups

Definitions:

- A **representation** of a group G on a $\mathbb{R}$- or $\mathbb{C}$-vector space $V \neq 0$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$ with the characteristic $\rho(gh) = \rho(g)\rho(h)$. It follows that $\rho(1) = \mathbb{1}$, $\varphi(g^{-1}) = \varphi(g)^{-1}$.
- The **regular representation** of finite group G is the representation onto the vector space $\mathbb{C}[G]$ of all the functions from G to $\mathbb{C}$: $(\rho_{\mathrm{reg}}(g)f)(h) = f(g^{-1}h),\ f \in \mathbb{C}[G],\ g, h \in G$.
- A **homomorphism of representations** $(\rho_1, V_1) \to (\rho_2, V_2)$ is a linear operation $\varphi : V_1 \to V_2$, so that $\varphi\rho_1(g) = \rho_2(g)\varphi$. $\mathrm{Hom}_G(V_1, V_2)$ is the space of all G-**equivariant** maps. If an invertible $\varphi \in \mathrm{Hom}_G(V_1, V_2)$ exists we have $(\rho_1, V_1) \cong (\rho_2, V_2)$.
- An **invariant subspace** of a (ρ, V) is a subspace $W \subset V$ such that $\rho(g)W \subset W\ \forall g \in G$. If $W \neq \{0\}$ we call the restriction $\rho|_W : G \to \mathrm{GL}(W)$ a **subrepresentation** of G.
- A representation (ρ, V) is **completely reducible**, if invariant subspaces $V_1, \ldots, V_n \subset V$ exist, such that $V = V_1 \oplus \cdots \oplus V_n$.

Useful Theorems:

- Let (ρ, V) be a finite-dimensional representation. If for every invariant subspace $W \subset V$ there exists another invariant subspace W' with $V = W \oplus W'$. Then (ρ, V) is completely reducible.
- Unitary representations $\rho(g) \in \mathrm{U}(n)$ are completely reducible.
- Let (ρ, V) be a representation of a finite group G. Then there exists a scalarproduct $(\cdot, \cdot)$ on V such tat (ρ, V) is unitary.
- Finite-dimensional complex representations of finite groups are completely reducible.
- Irreducible finite-dimensional complex representations of commutative groups are 1-dimensional.
- **Schur's Lemma**: Let $(\rho_1, V_1), (\rho_2, V_2)$ be irreducible complex representations of G.
  (i) $\varphi \in \mathrm{Hom}_G(V_1, V_2) \implies \varphi \equiv 0$ or φ is an isomorphism
  (ii) $\varphi \in \mathrm{Hom}_G(V_1, V_1)$. Then $\varphi = \lambda\mathbb{1},\ \lambda \in \mathbb{C}$

3 Representation Theory of Finite Groups

Definitions:

- The **character** $\chi_\rho : G \to \mathbb{C}$ of a finite-dimensional representation (ρ, V) of a group G is defined as $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. The representation may be written in any Basis of G.
  - $\chi_\rho(g) = \chi_\rho(hgh^{-1})$, e.g. χ_ρ is constant on conjugacy classes.
  - If $(\rho, V) \cong (\rho', V')$, then $\chi_\rho = \chi_{\rho'}$
  - $\chi_\rho(1) = \dim V$
  - $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$
  - $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)},\ \forall g \in G$
  - Choosing a basis $(\delta_g)_{g\in G}$ of delta functions where $\delta_g(g) = 1$ and 0 otherwise of $\mathbb{C}[G]$. It follows that $\rho_{\mathrm{reg}}(g)\delta_h = \delta_{gh}$ and $\chi_{\mathrm{reg}}(g) = |G|$ if $g = 1$ and 0 otherwise
- We define the scalar product $(f_1, f_2) = \frac{1}{|G|} \sum_{g\in G} \overline{f_1(g)} f_2(g)$
- A function $f : G \to \mathbb{C}$ is a **class function**, if $f(ghg^{-1}) = f(h)$.

Useful Theorems:

- A finite group G has a finite number of irreducible representations $(\rho_1, V_1), \ldots, (\rho_k, V_k)$. Let $d_\alpha = \dim(V_\alpha)$, then
  $$\textstyle\sum_{\alpha=1}^k d_\alpha^2 = |G|.$$
- Let $(\rho, V), (\rho', V')$ be irreducible unitary representations of G:
  $$\frac{1}{|G|} \sum_{g\in G} \overline{\rho_{ij}(g)} \rho_{kl}(g) = \frac{1}{\dim V} \delta_{ik}\delta_{jl}, \quad \forall i, j, k, l.$$
  If $(\rho, V) \ncong (\rho', V')$: $\frac{1}{|G|} \sum_{g\in G} \overline{\rho_{ij}(g)} \rho'_{kl}(g) = 0.$
- Let $\rho,\ \rho'$ be irreducible representations of a finite group G. Then $(\chi_\rho, \chi_{\rho'}) = 1$ if $(\rho, V) \cong (\rho', V')$ and 0 otherwise.
- Let $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ be the factorization of ρ into irreducible representations and σ be an irreducible representation. The number of ρ_i isomorphic to to σ is equal to (χ_ρ, χ_σ).
- Every irreducible representation of a finite group G is in the regular representation. If a representation has dimension d then it is in ρ_{reg} d times, $(\chi_\sigma, \chi_{\mathrm{reg}}) = d$. We can see that $\chi_{\mathrm{reg}}(g) = \sum_i d_i \chi_{\rho_i}(g)$.
- Let $\rho_1, \ldots, \rho_k$ be a list of non-isomorphic unitary representations. Let $\rho_{\alpha,ij},\ \alpha = 1, \ldots, k,\ 1 \leq i, j \leq d_\alpha$ be the matrix elements of ρ_α written in an orthonormal Basis. Then the functions $\sqrt{\dim V_\alpha}\rho_{\alpha,ij}$ build an orthonormal basis of $\mathbb{C}[G]$.
- Let G be a finite group. The characters $\chi_1, \ldots, \chi_k$ of the irreducible representations of G build an orthonormal basis of the Hilbert space of class functions.
- The number of irreducible complex finite-dimensional representations of a finite group G is equal to the number of conjugacy classes of G.
- Let $\rho_i : G \to \mathrm{GL}(V_i),\ 1 \leq i \leq k$ be a list of all non isomorphic irreducible representations of G. Let ρ be a representation on V. Let $V = U_1 \oplus \cdots \oplus U_n$ be a factorization of V into irreducible invariant subspaces. For every i, we define W_i as the direct sum of all U_j for which $\rho|_{U_j} \cong \rho_i$. The projection $p_i : V \to W_i$ is given by
  $$p_i(v) = \frac{\dim V_i}{|G|} \sum_{g\in G} \overline{\chi_i(g)} \rho(g) v$$
- The factorization $V = W_1 \oplus \cdots \oplus W_k$ is called **canonical factorization** and the subspaces W_i are called **isotypic components**.
- Let ρ be a finite-dimensional complex representation of a compact group G and $A : V \to V$ a diagonalizable linear function, with $\rho(g)A = A\rho(g),\ \forall g \in G$. Let $V = V_1 \oplus \cdots \oplus V_n$ be a factorization into irreducible representations. Then A has at most n different eigenvalues λ_k, each with multiplicity $\dim V_k$.

4 Die Drehgruppe und die Lorentzgruppe

Definitions:

- An **isometry** of the Euclidean space is a bijective function $f : \mathbb{R}^3 \to \mathbb{R}^3$, that conserves distances: $d(f(x), f(y)) = d(x, y),\ \forall x, y \in \mathbb{R}^3$. Therefore $v \in \mathbb{R}^n$ $A \in \mathrm{O}(n)$ exist, such that $f(x) = Ax + v\ \forall x \in \mathbb{R}^n$.
- The **Minkowski space** has the metric
  $$(x, y)_{1,3} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3.$$
- A vector is called **timelike**, if $(x, x)_{1,3} > 0$, **spacelike** if $(x, x)_{1,3} < 0$ and **lightlike** if $(x, x)_{1,3} = 0$ for $x \in \mathbb{R}^4$.

Useful Theorems:

- Let $\vec{n} \in \mathbb{R}^3$ be a unit vector and $O \in \mathrm{SO}(3)$ such that $O\vec{e}_3 = \vec{n}$. Then $R(\vec{n}, \theta) = OR_3(\theta)O^{-1}$ is the rotation around $\vec{n}$ by θ.
- Every $A \in \mathrm{SO}(3)$ can be written using **Euler angles** $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi]$ and $\psi \in [0, 2\pi)$ (uniquely if $\theta \neq 0, \pi$):
  $$A = R_3(\varphi) R_1(\vartheta) R_3(\psi)$$
  $$= \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & -\sin\theta\cos\varphi - \cos\psi\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ \cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & -\sin\theta\cos\varphi \\ \sin\psi\sin\theta & \cos\psi\cos\theta & \cos\theta \end{pmatrix}$$

Homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$:

- We define H_0 as the vectorspace of all hermitian 2×2 matrices, which have the form $\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$
- We define the scalar product $(X, Y) = \frac{1}{2}\mathrm{tr}(XY) = \frac{1}{2}\mathrm{tr}(XY^*)$
- We define the orthogonal representation $\varphi : \mathrm{SU}(2) \to \mathrm{GL}(H_0)$, $\varphi(A)X = AXA^{-1} = AXA^*$.
- The Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are an orthonormal basis of H_0
  $$\sigma_0 = \mathrm{Id},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
  since $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. Therefore we identify H_0 with $\mathbb{R}^3$.
- $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective with kernel $\{\pm\mathbb{1}\}$. This means $\mathrm{SU}(2)/\{\pm\mathbb{1}\} \cong \mathrm{SO}(3)$

The Lorentz-group:

- $\mathrm{O}(1, 3) = \{A \in \mathrm{GL}(4, \mathbb{R}) | (Ax, Ay)_{1,3} = (x, y)_{1,3},\ \forall x, y \in \mathbb{R}^4\}$
  For $A \in \mathrm{O}(1, 3)$: $AgA^\top = g, \quad g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
- A basis $b_0, \ldots, b_3$ of $\mathbb{R}^4$ is orthonormal, if $(b_i, b_j) = g_{ij},\ \forall i, j$.
- Let $b_0, \ldots, b_3$ and $b'_0, \ldots, b'_3$ be two orthonormal bases. Then there exists exactly one $A \in \mathrm{SO}(1, 3)$ such that $Ab_i = b'_i$.
- Lorentz transformation examples:
  $$\Lambda(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \quad L(\chi) = \begin{pmatrix} \cosh\chi & 0 & 0 & \sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix}$$
  The first is a rotation and the second is a Lorentz-boost in the x^3 direction with rapidity $\chi \in \mathbb{R}$.
  $$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
  Here P is a reflection of space and T is a reflection of time.
- For all Lorentz transformations A: $A^\top = PA^{-1}P = TA^{-1}T$.
- $\mathrm{O}_+(1, 3) = \{A \in O(1, 3) | A_{00} > 0\}$ is a subgroup of $\mathrm{O}(1, 3)$, which maps $Z_+ \to Z_+$, where Z_+ consists of all the **timelike** vectors with $x^0 > 0$. Transformations that preserve the direction of time are called **orthochronous**.
- $\mathrm{SO}_+(1, 3) = \{A \in \mathrm{O}_+(1, 3) |\det A = 1\}$ is called the special orthochronus Lorentz group.
- Every Lorentz transformation is in one of the following classes: $\mathrm{SO}_+(1, 3)$, $\{PX | X \in \mathrm{SO}_+(1, 3)\}$, $\{TX | X \in \mathrm{SO}_+(1, 3)\}$, $\{PTX | X \in \mathrm{SO}_+(1, 3)\}$.
- Every orthochronous transformation $A \in \mathrm{SO}_+(1, 3)$ is of the form $A = \Lambda(R_1)L(\chi)\Lambda(R_2),\ \chi \in \mathbb{R},\ R_1, R_2 \in \mathrm{SO}(3)$.
- $\mathrm{SO}_+(1, 3)$ is path connected: for each $A \in \mathrm{SO}_+(1, 3)$ there exists a continuous map $\gamma : [0, 1] \to M_3(\mathbb{R})$ for which $\gamma(t) \in \mathrm{SO}_+(1, 3) \forall t$ and $\gamma(0) = \mathbb{1}$, $\gamma(1) = A$.

Isomorphism $\mathrm{SL}(2, \mathbb{C})/\{\pm\mathbb{1}\} \to \mathrm{SO}_+(1, 3)$

- We define H as the 4-dimensional space of all the hermitian 2×2 matrices: every matrix has the form
  $$\hat{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x^0\sigma_0 + \sum_{j=1}^3 x^j\sigma_j$$
- $\forall x \in \mathbb{R}^4\ (x, x) = \det(\hat{x})$
- We define the representation $\varphi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(H)$, $\widehat{\varphi(A)x} = A\hat{x}A^*$. It follows that $\varphi(A) \in \mathrm{SO}_+(1, 3)$.
- The function $\varphi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}_+(1, 3)$ is a surjective homomorphism with kernel $\{\pm\mathbb{1}\}$. Therefore φ induces an isomorphism $\mathrm{SL}(2, \mathbb{C})/\{\pm\mathbb{1}\} \to \mathrm{SO}_+(1, 3)$. The restriction of φ onto $\mathrm{SU}(2)$ is the homomorphism above.

Lorentz group and the starry sky

- A **light beam** (through the origin) is a lightlike straight line $\{\lambda x, \lambda \in \mathbb{R}\},\ x \neq 0$ with $(x, x) = 0$. We define the set of all light beams as $S(K)$.
- The mapping $\Lambda \mapsto S_\Lambda$ is an isomorphism of $\mathrm{SO}_+(1, 3) \to S^2$. ($S_\Lambda$ describes where each light beam intersects with the surface of a ball with radius 1 around the origin.)
- Let $\Lambda \in \mathrm{SO}_+(1, 3)$. If we identify the set of light beams $S(K)$ with S^2, and using stereographic projection, with $\overline{\mathbb{C}}$, then S_Λ is a mobius transformation γ_A. The mapping $\Lambda \mapsto A$ is an isomorphism $\psi^{-1} : \mathrm{SO}_+(1, 3) \to \mathrm{SL}(2, \mathbb{C})$ and
  $$\psi(A) = \varphi(QAQ^{-1}), \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

5 Lie Groups and Lie Algebras

Lie Algebras:

- $\mathfrak{gl}(n,K) = M_n(K)$, $\mathfrak{gl}(V) = \mathrm{End}(V)$
- $\mathfrak{sl}(n,K) = \{X \in M_n(K) | \operatorname{tr} X = 0\}$
- $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) | X^* = -X\}$, $\dim = n^2$
- $\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) | X^* = -X,\ \operatorname{tr}(X) = 0\}$, $\dim = n^2 - 1$
- $\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) | X^\top = -X\}$ $(\operatorname{tr}(X) = 0)$, $\dim = (n^2 - n)/2$
- $\mathfrak{sp}(2n) = \{X \in Mat(2n, \mathbb{R} | X^\top J + JX = 0\}$

Definitions:

- **Lie groups** are groups G with the structure of a smooth manifold, such that inversion and group multiplication are continuous mappings (e.g. continuous symmetry groups).
- A **matrix-Lie-group** is a closed subgroup of $\mathrm{GL}(n,K)$.
- A **Lie algebra** $\mathfrak{g}$ is an $\mathbb{R}$- or $\mathbb{C}$-vectorspace with a bilinear Lie-bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ fulfilling $\forall X, Y, Z \in \mathfrak{g}$:
  - *bilinearity* $[\lambda X + Y, Z] = \lambda[X,Z] + [Y,Z]$
  - *antisymmetry* $[X,Y] = -[X,Y]$
  - *Jacobian identity* $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$
- We define the **Lie algebra** of a matrix-Lie-group G as:
$$\mathrm{Lie}(G) = \{X \in M_n(K) | \exp(tX) \in G\ \forall t \in \mathbb{R}\}.$$
with the **matrix commutator** $[X,Y] = XY - YX$.
- Let $M_n(K)$ be the vector space of all $n \times n$ matrices. Through the identification with K^{n^2}, $x = (x_{ij}) \mapsto (x_{11}, x_{12}, \dots, x_{nn})$ $M_n(K)$ gets the norm
$$\|x\| = \left(\sum_{i,j=1}^{n} |x_{ij}|^2\right)^{1/2} = (\operatorname{tr}(X^* X))^{1/2}$$
- $\|XY\| \le \|X\| \|Y\|$
- The **exponential** mapping $\exp : M_n(K) \to \mathrm{GL}(n,K)$
$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$
is invertible in the vicinity of 0. The inverse
$$\log X = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(X - \mathbb{1})^n}{n}$$
converges for $\|X - \mathbb{1}\| < 1$. For $X, Y \in M_n(\mathbb{C})$ we have:
  - If $XY = YX$: $\exp(X+Y) = \exp(X)\exp(Y)$
  - $\exp(X) \in \mathrm{GL}(n,\mathbb{C})$, $\exp(X)^{-1} = \exp(-X)$
  - $\forall A \in \mathrm{GL}(n,\mathbb{C})$: $A\exp(X)A^{-1} = \exp(AXA^{-1})$
  - $\det(\exp(X)) = \exp(\operatorname{tr}(X))$
  - $\exp(X^*) = \exp(X)^*$, $\exp(X^\top) = \exp(X)^\top$
- A mapping $X : \mathbb{R} \to \mathrm{GL}(n,K)$, $t \mapsto X(t)$ is called a **one-parameter group**, if it is continuously differentiable, $X(0) = \mathbb{1}$, and $\forall t, s \in \mathbb{R}$, $X(s+t) = X(s)X(t)$.
- X is called an **infinitesimal generator** of the one parameter group $t \mapsto \exp(tX)$.
- A **Lie algebra homomorphism** $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear mapping, so that $[\varphi(X), \varphi(Y)] = \varphi([X,Y])$. If φ is invertible it is an isomorphism and we have $\mathfrak{g}_1 \cong \mathfrak{g}_2$.

Useful Theorems:

- $\mathrm{Lie}(G)$ consists of all the tangential vectors $\dot{X}(0) = \frac{d}{dt} X(t)|_{t=0}$ of smooth curves $X :]-\varepsilon, \varepsilon[\to G$ with $X(0) = \mathbb{1}$ and $\varepsilon > 0$.
- Let $\varphi : G \to H$ be a differentiable homomorphism of matrix-Lie-groups. Then $\varphi_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ with
$$\varphi_*(X) = \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(tX)), \quad X \in \mathrm{Lie}(G)$$
is a Lie algebra homomorphism.
- **CBH** Let $X, Y \in M_n(K)$. For t small enough
$$\exp(tX)\exp(tY) = \exp\left(tX + tY + \frac{t^2}{2}[X,Y] + \mathcal{O}(t^3)\right)$$
- Let $G \subset \mathrm{GL}(n,K)$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and the mapping $\exp : \mathfrak{g} \to G$. Then there is an open neighborhood $0 \in U \subset \mathfrak{g}$ of 0 and an open neighborhood around $\mathbb{1} \in V \subset G$, so that $\exp : U \to \mathrm{GL}(n,K)$ is a smooth immersion with $\exp(U) = V$.
- Let $G \subset \mathrm{GL}(n,K)$ be a Lie Group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n,K)$. The group of all matrices of the form
$$\exp(X_1) \cdots \exp(X_k), \quad X_1, \dots, X_k \in \mathfrak{g},\ k \ge 1 <$$
is the connected component of $\mathbb{1} \in G$.

6 Representations of Lie Groups

Definitions:

- A **representation** (τ, V) of a Lie algebra $\mathfrak{g}$ is a linear map $\tau : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$, such that $[\tau(X), \tau(Y)] = \tau([X,Y])$.
- A polynomial $p(z_1, \dots, z_N)$ in N unknowns is called **homogeneous of degree** d if for $p(\lambda z_1, \dots, \lambda z_N) = \lambda^d p(z_1, \dots, z_N)$ $\forall \lambda \in \mathbb{R}$. They then are of the form $\sum_{|\alpha|=d} p_\alpha z^\alpha$.
- The **tensorproduct** of two finite dimensional representations (ρ, V), (ρ', V') of a group G is the representation $\rho \otimes \rho'$ onto $V \otimes V'$, given by $(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$. The corresponding representation $(\rho \otimes \rho')_*$ of $\mathfrak{g}$ is given by $(\rho \otimes \rho')_*(X) = \rho_*(X) \otimes \mathbb{1}_{V'} + \mathbb{1}_V \otimes \rho'_*(X)\ \forall X \in \mathfrak{g}$.

Useful Theorems:

- Let (ρ, V) be a representation of G and $X \in \mathrm{Lie}(G)$. Then
$$\rho_*(X) = \left.\frac{d}{dt}\right|_{t=0} \rho(\exp(tX)) \in \mathfrak{gl}(V)$$
is a representation of $\mathrm{Lie}(G)$, with
$$\rho(\exp tX) = \exp(t\rho_*(X)),\ t \in \mathbb{R},\ X \in \mathrm{Lie}(G)$$
- Every irreducible representation of $\mathrm{U}(1)$ is 1-dimensional and equivalent to $\rho_n : \mathrm{U}(1) \to \mathrm{GL}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, $z \mapsto z^n$ for a suitable $n \in \mathbb{N}$.
- Homogenous polynomials of degree d in N variables build a vector space. The monomials $z_1^{j_1} \cdots z_N^{j_N}$ with $j_k \ge 0$ and $\sum_k j_k = d$ are a basis of this vector space.
- Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a Lie group G. Then ρ maps one-parameter groups onto one-parameter groups.
- The restriction of a representation ρ of a Lie group G onto the one-component $G_0 \ni \mathbb{1}$ from G is uniquely defined by ρ_*.
- Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a connected Lie group G. Then ρ is irreducible iff ρ_* is irreducible.
- If $[X,Y] = 0$ for $\forall X, Y \in \mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{h}$ is a closed sub Lie algebra of $\mathfrak{g}$.
- (Clebsch-Gordan) The decomposition of a tensorproduct of irreductible representations of dimensions $n' + 1$, $n'' + 1$ is
$$\rho_{n'} \otimes \rho_{n''} \cong \rho_{n'+n''} \oplus \rho_{n'+n''-2} \oplus \cdots \oplus \rho_{|n'-n''|}.$$
The irreducible sub-representation of dimension $n' + n'' + 1 - 2l$ is spanned by $w_l, Fw_l, \dots, F^{n'+n''-2l}$, where
$$w_l = \sum_{j=0}^{l} (-1)^j \frac{(n'-j)!(n''-l+j)!}{j!(l-j)!} v'_j \otimes v''_{l-j}.$$
- For $A \in \mathrm{GL}(V)$, $B \in \mathrm{GL}(W)$ we have $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$, so the character of a tensorproduct of representations $\rho \otimes \rho'$ is $\chi \otimes \chi' = \chi \cdot \chi'$.

Irreducible representation of SU(2):

- Every $Z \in \mathfrak{sl}(n,\mathbb{C})$ can be uniquely written as $Z = X + iY$ with $X, Y \in \mathfrak{su}(n)$
- Let τ be a representation of $\mathfrak{su}(n)$ on V. Then
$$\tau_{\mathbb{C}}(X + iY) = \tau(X) + i\tau(Y)$$
defines a representation of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$
- $\tau_{\mathbb{C}}$ is irreducible iff τ is irreducible.
- A basis of $\mathfrak{sl}(2,\mathbb{C})$ is
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
- $[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$
- If $\tau : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ is a $\mathbb{C}$-linear representation, then
$$H = \tau(h), \quad E = \tau(e), \quad F = \tau(f)$$
fulfill the relation
$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H$$
- Let (τ, V) be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, $\lambda \in \mathbb{C}$ the eigenvalue of H with with the largest real part and v_0 an eigenvector to λ, so $Hv_0 = \lambda v_0$. It follows that $Ev_0 = 0$ and for $v_k = F^k v_0$ we have $Hv_k = (\lambda - 2k)v_k$ and $Ev_k = k(\lambda - k + 1)v_{k-1}$.
- Let $n = 0, 1, 2, \dots$ and $v_0, \dots, v_n$ be the standard basis of $V_n = \mathbb{C}^{n+1}$. Then
  - $Hv_k = (n - 2k)v_k$
  - $Ev_k = k(n + 1 - k)v_{k-1}$ (for $k = 0$: $Ev_0 = 0$)
  - $Fv_k = v_{k+1}$ (for $k = n$: $Fv_n = 0$)

  defines an irreducible representation τ_n of $\mathfrak{sl}(2,\mathbb{C})$. Every complex $(n+1)$-dimensional irreducible representation from $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to τ_n.
- For every $n = 0, 1, 2, \dots$ there exists (up to isomorphism) exactly one irreducible representation (ρ_n, U_n) of SU(2) of dimension $n + 1$.
$$U_n = \{\sum_{j=0}^{n} c_j z_1^j z_2^{n-j} | c_j \in \mathbb{C}\}$$
is the space of homogeneous polynomials of degree n in two variables. For $A \in \mathrm{SU}(2)$, $f \in U_n$
$$(\rho_n(A)f)(z) = f(A^{-1}z)$$
ρ_n is unitary with respect to the dot product where
$$v_k := \frac{z_1^k z_2^{n-k}}{\sqrt{k!(n-k)!}}$$
is an orthonormal basis.
- (ρ_n, U_n) or any equivalent is called Spin-$\frac{1}{2}$-representation.
- $(\rho_n)_* = \tau_n : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^{n+1})$, where we use the isomorphism $\mathbb{C}^{n+1} \cong U_n$, $v_k \mapsto p_k := \frac{(-1)^k}{(n-k)!} z_1^k z_2^{n-k}$.
- For $n \in \mathbb{N}_0$ even, there exists (up to isomorphism) exactly one irreducible representation $(\tilde{\rho}_n, U_n)$ of SO(3) with $\dim U_n = n + 1$ (uneven):
$$(\tilde{\rho}_n(R)p)(z) = p(A^{-1}z),$$
where $R \in SO(3)$ and $\varphi(A) = R$, with φ being the $\mathrm{SU}(2) \to \mathrm{SO}(3)$ double cover.

Harmonic Polynomials and Spherical Harmonics

- We define $H_l = \{\sum_{|\alpha|=l} c_\alpha x^\alpha,\ c_\alpha \in \mathbb{C}\}$ with $\dim H_l = \frac{1}{2}(l+1)(l+2)$ as the space of homogeneous polynomials of order l in 3 variables. If $P(x) \in H_l$ then $P(R^{-1}x) \in H_l\ \forall R \in \mathrm{SO}(3)$.
- We have the representation of SO(3) on H_l:
$$(\rho(R)f)(x) = f(R^{-1}x)$$
- We define a dot product on H_l such that ρ is unitary:
$$(f,g) = \int_{|x|=1} \overline{f(x)} g(x)\, \mathrm{d}\Omega(x)$$
- The Laplace operator $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ maps H_l onto H_{l-2}.
- We define the harmonic polynomials in H_l as
$$V_l = \{f \in H_l | \Delta f = 0\}$$
with $\dim V_l \ge \dim H_l - \dim H_{l-2} = 2l + 1$ and
$$H_l = \bigoplus_{k=0}^{\lfloor l/2 \rfloor} r^{2k} V_{l-2k}$$
- We can now define a representation ρ of SU(2):
$$(\rho(A)u)(x) = \mathfrak{u}(\varphi(A)^{-1}x), \quad A \in SU(2),\ u \in V_l,$$
where $\varphi(\exp\left(-i\sum_{j=1}^{3} \sigma_j n_j \theta/2\right)) = R(n,\theta)$, $|n| = 1$.
- The corresponding Lie-algebra representation of $\mathfrak{su}(2)$ onto $\mathfrak{gl}(V_l)$ is:
$$(\tau(X)u)(x) = -2\sum_{\beta=1}^{3} (\alpha \wedge x)_\beta \frac{\partial u}{\partial x_\beta}(x),$$
where $X = \sum_j (\alpha_j(-i\sigma_j) = \left(\begin{smallmatrix} -i\alpha_3 & -i\alpha_1 - \alpha_2 \\ i\alpha_1 + \alpha_2 & i\alpha_3 \end{smallmatrix}\right) \in \mathfrak{su}(2)$ and $\alpha = n\theta/2 \in \mathbb{R}^3$.
- h, e, f correspond to $\alpha = (0,0,i)$, $\alpha = (\frac{i}{2}, -\frac{1}{2}, 0)$, $\alpha = (\frac{i}{2}, \frac{1}{2}, 0)$, so $H = \tau(h),\ E = \tau(e),\ F = \tau(f)$:
$$H = -2i\left(x_i \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right),$$
$$E = x_3\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) - (x_1 + ix_2)\frac{\partial}{\partial x_3},$$
$$F = x_3\left(-\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) + (x_1 - ix_2)\frac{\partial}{\partial x_3}$$
- u_{lm} recursively defined by $u_{ll}(x) = \sqrt{\frac{(2l+1)!}{4\pi}} \frac{(-1/2)^l}{l!}(x_1 + ix_2)^l$ and $u_{l,l-j} = \frac{Fu_{l,l-j+1}(x)}{\sqrt{j(2l+1-j)}}$, defines an orthonormal basis with:
$$Hu_{lm} = 2mu_{lm}$$
$$Eu_{lm} = \sqrt{(l-m)(l+m+1)}u_{l,m+1}$$
$$Fu_{lm} = \sqrt{(l-m+1)(l+m)}u_{l,m-1}$$
- A **spherical harmonic** (**Kugelfunktion**) $Y : S^2 \to \mathbb{C}$ of index l is the restriction onto $S^2 \subset \mathbb{R}^3$ of a homogeneous harmonic polynomial of degree l.
- Let $\hat{V}_l$ be the space of all spherical harmonics of index l. We have: $\quad Y = Y(\theta,\varphi) \in \hat{V}_l \iff r^l Y(\theta,\varphi) \in V_l$.
- $Y_{lm}(\theta,\varphi) = r^{-l} u_{lm}(r,\theta,\varphi)$ is an orthonormal basis of $\hat{V}_l$.
- $Y_{lm}(\theta,\varphi)$ is an eigenvector of Δ_{S^2}: $\quad \Delta_{S^2} Y_{lm} = -l(l+)Y_{lm}$