

# 02 - Moment of Inertia

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## Abstract

The spring constant  $D$  of a spring was determined. Further, Steiner's theorem was verified experimentally. Lastly, it was shown that the moment of inertia of a dumbbell of different axes of rotation is an ellipse. Accounting the errors, the results gathered appear to be reasonable. Both the verification of Steiner's theorem and the ellipse of inertia yielded expected results when looking at the respective graphs.

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# 1 Introduction

When a rigid body is rotated around a fixed axis, there will be a circular motion where each point has the same angular velocity. This is due to the fact that the distance from the fixed axis to any point of mass of the body is constant. With the moment of inertia  $\theta = \sum \mu r^2$  and angular momentum  $M$ , we get the law of motion of the rigid body:

$$M = \theta \frac{d^2\phi}{dt^2}.$$

Further, using Steiner's theorem, we can determine the moment of inertia  $\theta_A$  of any axis that is displaced by  $d$  from the parallel axis going through the center of mass. It is calculated using the moment of inertia  $\theta_S$  of the parallel axis going through the center of mass and the object's mass  $m$  as follows:

$$\theta_A = \theta_S + d^2m.$$

The equation of motion of a torsion pendulum is given by

$$M = \theta \frac{d^2\phi}{dt^2} = -D\phi.$$

Using this, we can determine the oscillation period  $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\theta}{D}}$ . If  $\theta$  is the moment of inertia of a mass and  $\theta'$  of the torsion axis, we have  $T' = \pi\sqrt{\frac{\theta+\theta'}{D}}$ , which gives us the torsion coefficient  $D = \frac{4\pi^2\theta'}{T'^2 - T^2}$ .

# 2 Experiment

The experiment consisted of three parts. The materials used for them were a scale, a ruler, a stopwatch, a circular disc, two beams, a dumbbell, and the apparatus with a spring shown in figure 1.

**Part 1: Determination of the spring constant  $D$**  We determined the spring constant  $D$  of the torsion axis. For this, two beams were set up on the torsion axle perpendicular to the principle axis of said beams. We then measured the oscillation period

$$T_1 = 2\pi\sqrt{\frac{\theta_K + 2\theta_0}{D}}.$$

After that, we added a circular disc of mass  $M_S$  and with radius  $R_S$  and measured the new oscillation period

$$T_2 = 2\pi\sqrt{\frac{\theta_K + 2\theta_0 + \theta_S}{D}}$$



Figure 1: Setup used in the experiment. For the first two parts, the dumbbell was replaced with two beams (and a disc) (source: [1])

In both instances,  $\theta_K$  is the moment of inertia of the holder and the torsion axis,  $\theta_0$  is the moment of inertia of one beam with respect to a rotation axis that passes through its center of mass, and  $\theta_S$  is the moment of inertia of the disc in relation to a principle axis perpendicular to its surface. Using  $\theta_S = \frac{1}{2}M_S R_S^2$ , we then calculated

$$D = \frac{4\pi^2\theta_S}{T_2^2 - T_1^2}.$$

**Part 2: Validity of Steiner's principle** We verified Steiner's principle. The beams used in the previous setup were shifted outwards by a distance  $d$  step by step in increments of 5cm in opposite directions as shown in figure 2. The oscillation period was measured each time, with which we determined the angular momentum  $\theta_A$  of a single beam in each setup using

$$\theta_A = \frac{T_A^2 D}{8\pi^2}$$

Using the theoretical value for the moment of inertia of a cuboid of mass  $M$  and edges of lengths  $a, b$  and  $c$ , which rotates around a principle axis parallel to  $a$ , given by

$$\theta_0 = M \frac{b^2 + c^2}{12},$$

and Steiner's theorem, we can calculate theoretical values for the moment of inertia of a beam in each setup:

$$\theta_{A,\text{theor.}} = \theta_0 + d^2 M$$

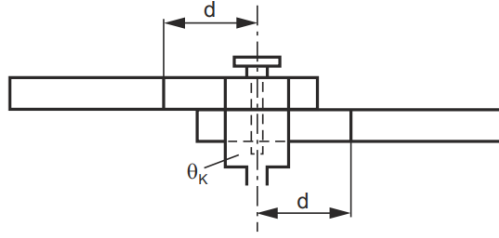


Figure 2: Setup for the verification of Steiner's theorem (source:[2])

**Part 3: The ellipse of inertia** We determined the moments of inertia of a dumbbell for different axes of rotation lying in a plane. Furthermore, we showed that the oscillation period and the cosine of the angle traces an ellipse. For this, we fit a weight with a scale ring on the axis of rotation as shown in figure 1. Then, we rotated the weight to tilt angles between  $0^\circ$  and  $90^\circ$  in increments of  $10^\circ$  each and measured the oscillation period for each tilt angle.

### 3 Results

**Part 1: Determination of the spring constant  $D$**  For the period  $T_1$  of the two beams we measured a value of  $T_1 = (6.48 \pm 0.17)$  s. The period of the system after adding the disc of mass  $M_S = (3.0800 \pm 0.0005)$  kg and radius  $R_S = (11.80 \pm 0.05)$  cm was  $T_2 = (8.04 \pm 0.13)$  s. We then calculated the moment of inertia of the disc and received a value of  $\theta_S = (0.0241 \pm 0.0002)$  kgm<sup>2</sup>. Through these values we finally obtained the spring constant,  $D = (0.037 \pm 0.005)$  kgm<sup>2</sup>/s<sup>2</sup>.

**Part 2: Validity of Steiner's theorem** A single beam had a mass of  $M = (0.611 \pm 0.001)$  kg, width of  $b = (0.0100 \pm 0.0005)$  m and a length of  $c = (0.6000 \pm 0.0005)$  m. The measured values for  $T_A$  and the consequently calculated moments of inertia  $\theta_A$  of a single beam displaced by distance  $d$ , along with the theoretical values  $\theta_{A,\text{theor.}}$  are shown in table 1. The same data is plotted in figure 3 with a linear fit applied to the experimental results.

**Part 3: The ellipse of inertia** The values obtained in this part can be seen in table 2. Figure 4 further shows the relation between  $\cos^2 \phi$  and  $T_s^2$ .

$d[\text{cm}]$	$T_A[\text{s}]$	$\theta_A [\text{kgm}^2]$	$\theta_{A,\text{theor.}} [\text{kgm}^2]$
0	$6.54 \pm 0.05$	$0.020 \pm 0.003$	$0.0183 \pm 0.00009$
5	$6.77 \pm 0.05$	$0.022 \pm 0.003$	$0.0199 \pm 0.0001$
10	$7.56 \pm 0.07$	$0.027 \pm 0.004$	$0.0244 \pm 0.0002$
15	$8.43 \pm 0.11$	$0.034 \pm 0.005$	$0.0321 \pm 0.0002$
20	$9.91 \pm 0.11$	$0.047 \pm 0.006$	$0.0428 \pm 0.0003$
25	$11.2 \pm 0.13$	$0.060 \pm 0.008$	$0.0565 \pm 0.0003$

Table 1: Measurements gathered for the verification of Steiner's theorem.  $T_a$  is the period of the oscillation,  $\theta_A$  is the moment of inertia determined from our measurements and  $\theta_{A,\text{theor.}}$  is theoretically calculated value.

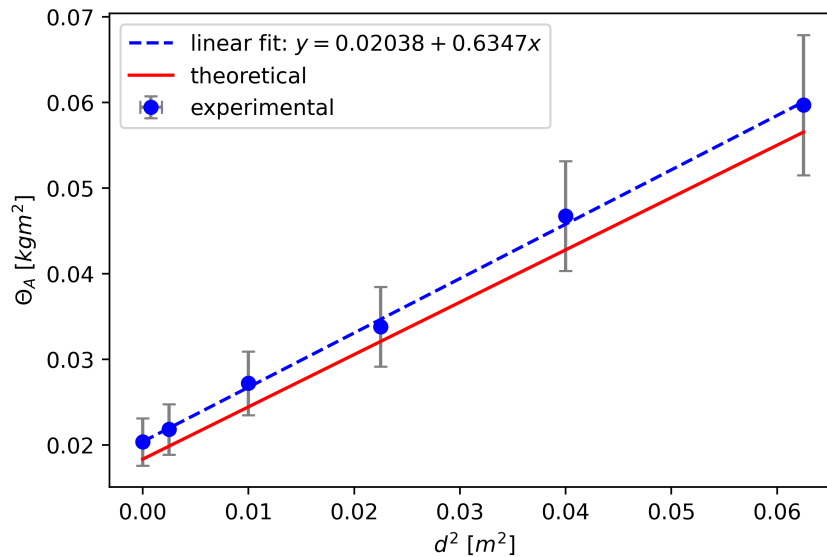


Figure 3: Moment of inertia  $\theta_A$  of two identical beams displaced oppositely by  $d$  from the rotational axis. The displacement of the theoretical and experimental results, occurs because the moment of inertia of the torsion axis mount was neglected in the theoretical calculation.

$\phi$	$T_s$ [s]	$T_s^2$ [s <sup>2</sup> ]	$\cos^2 \phi$
0°	4.88 ± 0.09	23.8 ± 0.9	1.00 ± 0.00
10°	4.81 ± 0.07	23.1 ± 0.7	0.96 ± 0.05
20°	4.66 ± 0.08	21.7 ± 0.7	0.88 ± 0.10
30°	4.43 ± 0.17	19.6 ± 1.5	0.75 ± 0.13
40°	4.16 ± 0.15	17.3 ± 1.2	0.59 ± 0.16
50°	3.92 ± 0.19	15.3 ± 1.5	0.41 ± 0.16
60°	3.73 ± 0.10	13.9 ± 0.7	0.25 ± 0.14
70°	3.64 ± 0.05	13.2 ± 0.4	0.12 ± 0.10
80°	3.46 ± 0.13	11.9 ± 0.9	0.03 ± 0.05
90°	3.44 ± 0.06	11.8 ± 0.4	0.00 ± 0.00

Table 2: Measurements gathered for the ellipse of inertia.  $\phi$  is the tilt angle of the dumbbell and  $T_s$  is the period of the system.

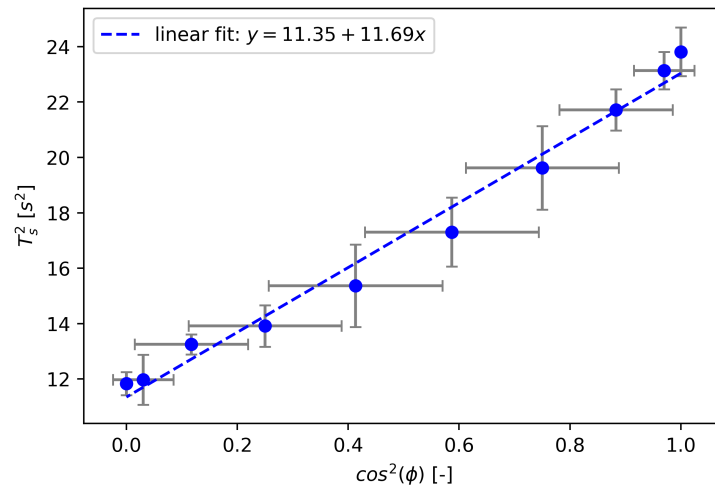


Figure 4: The square of the oscillation period  $T_s^2$  of a dumbbell on a torsion pendulum angled by  $\phi$  plotted against  $\cos^2 \phi$ . The linear dependence shows that the graph of  $T_s$  and  $\cos \phi$  forms an ellipse.

## 4 Data analysis

The errors in measurements stem mostly from the precision of the scale and the ruler used in the experiment. An additional error occurred due to the reaction time of the person measuring the period with help of a stopwatch, which was approximated by taking the standard deviation of three trials of each measurement. In order to determine the errors of the calculated values, Gauss error propagation was used. The formulas can be found in the [appendix](#).

## 5 Discussion

In the first part we determined the spring constant  $D$  accurate enough, such that Steiner's theorem and the ellipse of inertia could be confidently shown.

As the plot in figure 3 shows, the theoretical graph and the linear fit of our experimental data are parallel, which shows that Steiner's theorem is in fact true. Though a small vertical displacement between the two lines can be seen. This is explained by the moment of inertia of the torsion axis mount, which was neglected in the theoretical calculation. Therefore, we can calculate its moment of inertia to be  $M_{\text{tam}} \approx 0.0020 \text{ kgm}^2$ , which is simply the vertical difference of the two lines in the plot.

Finally, in the third part, as the plot in figure 4 is linear, we showed that the graph of the oscillation period  $T_s$  and the cosine of the angle  $\cos \varphi$  form an ellipse.

## 6 Conclusion

Firstly, we accurately determined the spring constant to be  $D = (0.037 \pm 0.005) \text{ kgm}^2/\text{s}^2$ , by measuring the oscillation period of two setups. First of which had an unknown moment of inertia while in the second one an object with a known moment of inertia was added.

Secondly, we showed that Steiner's theorem holds true by directly measuring the moment of inertia of different setups. The measured values scaled equally to what Steiner's theorem predicts. A small offset was recorded, which is explained by the added moment of inertia of the torsion axis mount.

Finally, a plot of the oscillation period of a dumbbell tilted at various angles shows that the oscillation period and the cosine of the tilt angle trace an ellipse.

## References

- [1] <https://ap.phys.ethz.ch/02.html>
- [2] [https://ap.phys.ethz.ch/Anleitungen/Bilingual/02\\_Manual.pdf](https://ap.phys.ethz.ch/Anleitungen/Bilingual/02_Manual.pdf)

## 7 Appendix

### Gauss error propagation formulas

$$\begin{aligned} \Delta\theta_S &= \sqrt{\frac{1}{2}R_S^2 \cdot \Delta M_S^2 + (M_S R_S)^2 \cdot \Delta R_S^2} \\ \Delta D &= \sqrt{\left(\frac{4\pi^2}{T_2^2 - T_1^2}\right)^2 \cdot \Delta\theta_S^2 + \left(\frac{8\pi^2\theta_S T_2}{(T_2^2 - T_1^2)^2}\right)^2 \cdot \Delta T_2^2 + \left(\frac{8\pi^2\theta_S T_1}{(T_2^2 - T_1^2)^2}\right)^2 \cdot \Delta T_1^2} \\ \Delta\theta_A &= \sqrt{\left(\frac{T_A D}{4\pi^2}\right)^2 \cdot \Delta T_A^2 + \left(\frac{T_A^2}{8\pi^2}\right)^2 \cdot \Delta D^2} \\ \Delta\theta_0 &= \sqrt{\left(\frac{b^2 + c^2}{12}\right)^2 \cdot \Delta M^2 + \left(\frac{Mb}{6}\right)^2 \cdot \Delta b^2 + \left(\frac{Mc}{6}\right)^2 \cdot \Delta c^2} \\ \Delta\theta_{A,\text{theor.}} &= \sqrt{2^2 \cdot \Delta\theta_0 + (4dm)^2 \cdot \Delta d^2 + (2d^2)^2 \cdot \Delta m^2} \\ \Delta T_A &= \sqrt{\left(\frac{\sqrt{2}\pi}{D\sqrt{\frac{\theta_A}{D}}}\right)^2 \cdot \Delta\theta_A^2 + \left(\frac{\sqrt{2}\pi\theta_A}{D^2\sqrt{\frac{\theta_A}{D}}}\right)^2 \cdot \Delta D^2} \\ \Delta(T_s^2) &= \sqrt{(2T_s)^2 \cdot \Delta T_s^2} = 2T_s \cdot \Delta T_s \\ \Delta(\cos^2 \phi) &= \sqrt{(2 \cos \phi \sin \phi)^2 \cdot \Delta\phi^2} \end{aligned}$$

### Proofs

To be proven:

$$\theta_0 = M \frac{b^2 + c^2}{12}$$

The moment of inertia of a rigid body is  $\int \rho x^2 dx$ . Since  $\rho = \frac{M}{V}$ , we get for our case:

$$\begin{aligned} \frac{M}{abc} \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (x^2 + y^2) dz dy dx &= \frac{M}{abc} \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} a(x^2 + y^2) dz dy \\ &= \frac{M}{abc} \int_{-c/2}^{c/2} \frac{a(12bx^2 + b^3)}{12} dz \\ &= \frac{M}{abc} \frac{a(cb^3 + c^3b)}{12} \\ &= M \frac{b^2 + c^2}{12} \quad \text{q.e.d.} \end{aligned}$$

To be proven:  $\theta_S = \frac{1}{2}M_S R_S^2$  Analog to before, we get:

$$\begin{aligned}\frac{M_S}{\pi R_S^2} \int_0^{2\pi} \int_0^{R_S} r^2 \cdot r dr d\phi &= \frac{M_S}{\pi R_S^2} \int_0^{2\pi} \frac{R_S^4}{4} d\phi \\ &= 2\pi \frac{M_S}{\pi R_S^2} \frac{R_S^4}{4} \\ &= \frac{1}{2} M_S R_S^2 \quad q.e.d.\end{aligned}$$

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Name 2: Sajoni Muganyizi ..... Platz Nr: 3 .....

<b>2</b>	<b>Trägheitsmoment</b>
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1. Bestimmung von D:  $m_{\text{Stange}} = 611 \text{ g} \pm 1 \text{ g}$

$T_1 = \underline{6.50, 6.42, 6.32, 6.50, 6.61}$  ( $T_1$ ) =  $6.484 \pm 0.17 \text{ s}$      $T_2 = \underline{8.03, 7.87, 8.22, 8.06, 8.00}$  ( $T_2$ ) =  $8.036 \pm 0.15 \text{ s}$

$M_S = 3.08 \pm 0.0005 \text{ kg}$      $R_S = 11.8 \pm 0.05 \text{ cm}$      $\Theta_S = 0.0214 \pm 0.0002 \text{ kgm}^2$

$D = \underline{0.037 \pm 0.005} \frac{\text{kgm}^2}{\text{s}^2}$

2. Satz von Steiner:  $M = 611 \pm 1 \text{ g}$  (ein Balken)     $b = 1.0 \pm 0.05 \text{ cm}$      $c = 6.0 \pm 0.05 \text{ cm}$

	2 Balken	1 Balken	1 Balken
d	$T_a$	$\Theta_A^{(\text{exp})}$	$\Theta_A^{(\text{ber})}$
0 cm	(6.55, 6.53, 6.53, 6.54) $6.54 \pm 0.005$	$\Theta_0 = 0.020 \pm 0.003$	$\Theta_0 = 0.020 \pm 0.0004$
5 cm	(6.78, 6.82, 6.72, 6.77) $6.77 \pm 0.005$	$0.022 \pm 0.003$	$0.021 \pm 0.0005$
10 cm	(7.55, 7.56, 7.63, 7.56) $7.56 \pm 0.07 \text{ s}$	$0.027 \pm 0.004$	$0.026 \pm 0.0005$
15 cm	(8.53, 8.44, 8.32, 8.45) $8.45 \pm 0.11 \text{ s}$	$0.034 \pm 0.005$	$0.035 \pm 0.0006$
20 cm	(9.94, 9.78, 10.00, 9.91) $9.91 \pm 0.11 \text{ s}$	$0.047 \pm 0.006$	$0.046 \pm 0.0007$
25 cm	(11.15, 11.09, 11.39, 11.22) $11.22 \pm 0.15 \text{ s}$	$0.060 \pm 0.008$	$0.061 \pm 0.0008$

Grafische Darstellung:  $\Theta_A = f(d^2)$  !

3. Trägheitsellipse:

φ ± 0.5°	T <sub>S</sub>	(T <sub>S</sub> ) <sup>2</sup>	COS <sup>2</sup> φ
0°	(4.91, 4.94, 4.78, 4.81) $4.81 \pm 0.005$	$23.1 \pm 0.6 \text{ s}^2$	$1.00 \pm 0.00$
10°	(4.75, 4.88, 4.79, 4.81) $4.81 \pm 0.005$	$23.1 \pm 0.5 \text{ s}^2$	$0.96 \pm 0.05$
20°	(4.75, 4.60, 4.62, 4.66) $4.66 \pm 0.005$	$21.7 \pm 0.3 \text{ s}^2$	$0.86 \pm 0.10$
30°	(4.62, 4.35, 4.31, 4.32) $4.32 \pm 0.005$	$18.6 \pm 1.1 \text{ s}^2$	$0.75 \pm 0.13$
40°	(4.03, 4.06, 4.54, 4.16) $4.16 \pm 0.005$	$17.3 \pm 0.3 \text{ s}^2$	$0.53 \pm 0.16$
50°	(3.93, 3.72, 4.10, 3.92) $3.92 \pm 0.005$	$15.3 \pm 1.1 \text{ s}^2$	$0.41 \pm 0.16$
60°	(3.79, 3.78, 3.62, 3.75) $3.75 \pm 0.005$	$14.0 \pm 0.5 \text{ s}^2$	$0.25 \pm 0.19$
70°	(3.59, 3.69, 3.65, 3.64) $3.64 \pm 0.005$	$13.2 \pm 0.3 \text{ s}^2$	$0.12 \pm 0.10$
80°	(3.40, 3.37, 3.60) $3.40 \pm 0.15 \text{ s}$	$11.6 \pm 0.6 \text{ s}^2$	$0.03 \pm 0.05$
90°	(3.38, 3.50, 3.90) $3.44 \pm 0.06 \text{ s}$	$11.8 \pm 0.3 \text{ s}^2$	$0.00 \pm 0.00$

Grafische Darstellung:  $(T_S)^2 = f(\cos^2 \varphi)$

Figure 5: Protocol sheet