Moment of Inertia

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Abstract

We showed that our pendulum has a period of $T_0 = (1.55 \pm 0.01)\,\mathrm{s}$, which is right between the period predicted by a mathematical and a physical pendulum: $T_{0,\mathrm{math}} = (1.528 \pm 0.023)\,\mathrm{s}$, $T_{0,\mathrm{phys}} = (1.600 \pm 0.024)\,\mathrm{s}$. Secondly we analyzed the coupling behavior of a coupled pendulum by measuring the period of a symmetric τ_ω and anti-symmetric case τ_Ω. Using these measurements we then calculated the period of a wandering oscillation, where one pendulum starts at a certain displacement, while the other starts at equilibrium. Further we calculate the period of the beating that occurs. Both these calculations we then compared to our measurements. Further we analyzed the coupling itself both by calculating the coupling torque D_f and the degree of coupling k dynamically, using the values from the previous experiment, and statically, by looking at the ratio between the displacement of one pendulum Φ_1 caused by the displacement of the other pendulum Φ_2. Finally we did a video analysis of a coupled oscillation and plotted the angles of both pendulums against the time.

Contents

1 Introduction

In this lab we observed the different ways a coupled pendulum oscillates. The equation of motion of a pendulum that is free to rotate and oscillate around a fixed axis a under the effect of gravity is given by $\theta\ddot{\varphi} = M = -mgl\sin\varphi$, where θ is the moment of inertia of the pendulum, m its mass, l the distance between the center of mass S and the axis a, φ is the angle from equilibrium, and M is the restoring torque of the pendulum. If we substitute the directional moment $D_g = mgl$ and use small angle approximation for $\varphi < 7°$ we obtain the equation for the oscillation

$$\ddot{\varphi} + \frac{D_g}{\theta}\varphi = 0,$$

which has the solution $\varphi(t) = A\sin(\omega_0 t + \delta)$, with the angular frequency $\omega_0 = \sqrt{D_g/\theta} = 2\pi/T_0$ and phase δ. It follows that the oscillation period is given by

$$T_0 = 2\pi\sqrt{\theta/D_g}. \tag{1}$$

In the ideal case of a mathematical pendulum, where the entire mass of the pendulum is concentrated in the center of mass, we have the equation

$$T_0 = 2\pi\sqrt{l/g}. \tag{2}$$

In the first part of the experiment, we will be measuring the oscillation period of a pendulum and comparing it to the oscillation period of a mathematical pendulum.
In the case of two identical coupled pendulum, we have an extra coupling torque, which depends on the difference of the two deflection angles $D_f(\varphi_2 - \varphi_1)$. We can set up the equation of motion for each pendulum

$$\theta\ddot{\varphi}_1 = D_g\varphi_1 + D_f(\varphi_2 - \varphi_1)$$
$$\theta\ddot{\varphi}_2 = D_g\varphi_2 - D_f(\varphi_2 - \varphi_1).$$

By solving for $(\varphi_2 + \varphi_1)$ and $(\varphi_2 - \varphi_1)$, we see that both equations represent undamped oscillations

$$(\varphi_2 + \varphi_1) = 2A\cos(\omega t + \delta)$$
$$(\varphi_2 - \varphi_1) = 2B\cos(\Omega t + \Delta),$$

with the angular frequencies $\omega = \sqrt{D_g/\theta}$ and $\Omega = \sqrt{(D_g + 2D_f)/\theta}$. We can solve in order to get the equation for each individual φ

$$\varphi_1 = A\cos(\omega t + \delta) - B\cos(\Omega t + \Delta)$$
$$\varphi_2 = A\cos(\omega t + \delta) + B\cos(\Omega t + \Delta)$$

At $t = 0$, if we have the initial conditions

$$\begin{cases} \varphi_1 = \Phi, \quad \varphi_2 = \Phi \\ \dot{\varphi}_1 = \dot{\varphi}_2 = 0, \end{cases}$$

which is a symmetric oscillation, then the oscillation of the system is represented by $\varphi_1 = \varphi_2 = \Phi \cos \omega t$ and the period of oscillation is

$$\tau_\omega = 2\pi/\omega = 2\pi\sqrt{\theta/D_g}. \tag{3}$$

If we have the starting condition

$$\begin{cases} \varphi_1 = -\Phi, \quad \varphi_2 = \Phi \\ \dot{\varphi}_1 = \dot{\varphi}_2 = 0, \end{cases}$$

or an antisymmetric oscillation, then the equations of oscillation are $\varphi_2 = -\varphi_1 = \Phi \cos \Omega t$ with oscillation period

$$\tau_\Omega = 2\pi/\Omega = 2\pi\sqrt{\theta/(D_g + 2D_f)}. \tag{4}$$

If we only displace one pendulum at $t = 0$, we have the initial conditions

$$\begin{cases} \varphi_1 = 0, \quad \varphi_2 = \Phi \\ \dot{\varphi}_1 = \dot{\varphi}_2 = 0. \end{cases}$$

The equations of oscillation are in this case

$$\varphi_1 = \Phi \sin\left(\frac{\Omega + \omega}{2} t\right) \sin\left(\frac{\Omega - \omega}{2} t\right)$$
$$\varphi_2 = \Phi \cos\left(\frac{\Omega + \omega}{2} t\right) \cos\left(\frac{\Omega - \omega}{2} t\right).$$

The frequency $\frac{\Omega+\omega}{2}$ corresponds to an oscillation with period

$$\tau = \frac{4\pi}{\Omega + \omega}$$

and the period of the beat is given by

$$T_S = \frac{2\pi}{\Omega - \omega}.$$

To calculate the theoretical values for τ and T_S we have the folowing equations

$$\frac{1}{\tau} = \frac{1}{2}\left(\frac{1}{\tau_\omega} + \frac{1}{\tau_\Omega}\right) \tag{5}$$
$$\frac{1}{T_S} = \frac{1}{\tau_\Omega} - \frac{1}{\tau_\omega}. \tag{6}$$

There are two methods to calculate the coupling moment D_f and the degree of coupling k, a static and dynamic method. To calculate them dynamically, the equations are

$$D_f = 2\pi^2\theta\left[\frac{1}{\tau_\Omega^2} - \frac{1}{\tau_\omega^2}\right] \tag{7}$$

$$k = \frac{D_f}{D_g + D_f}. \tag{8}$$

For the static method, we displace one pendulum by an angle Φ_1 and then measure the displacement the other pendulum experiences Φ_2 and then calculate the coupling moment and degree of coupling using

$$D_f = gl\left(m + \frac{m'}{2}\right)\frac{\Phi_1}{\Phi_2 - \Phi_1} \tag{9}$$

$$k = \frac{\Phi_1}{\Phi_2} \tag{10}$$

The moment of inertia about an axis through the center of mass of a cylinder is given by $\theta_S = \frac{m}{12}(3r^2 + h^2)$. In our case, both the shaft and the mas attached are both cylindrical in shape, so using the Parallel Axis theorem, we could calculate the moment of inertia for the pendulum in our setup.

2 Experiment

Before conducting any experiments, we measured the the masses and the dimensions of all the components of the double pendulum. The setup consisted of two pendulums as shown in figure 1. These were connected by a spring, which was attached to both shafts.

Physical Pendulum Firstly we removed the spring in order to measure the period of single pendulum. We did this by letting the pendulum go at a certain amplitude with minimal initial velocity, and then measuring the time it took to complete 50 oscillations. We then compared our measurements with theoretical values calculated with the equations (2) and (1) of a mathematical and a physical pendulum respectively.

Coupled Oscillations Secondly we looked at various scenarios of coupled oscillations. These were:

- Symmetric: Both pendulums are displaced by the same angle $\varphi_1 = \varphi_2 = \Phi$.
- Anti-symmetric: Both pendulums are displaced by opposite angles $\varphi_2 = -\varphi_1 = \Phi$.
- Wandering: *One* pendulum starts with a displacement, while the other starts at equilibrium $\varphi_1 = \Phi, \varphi_2 = 0$.

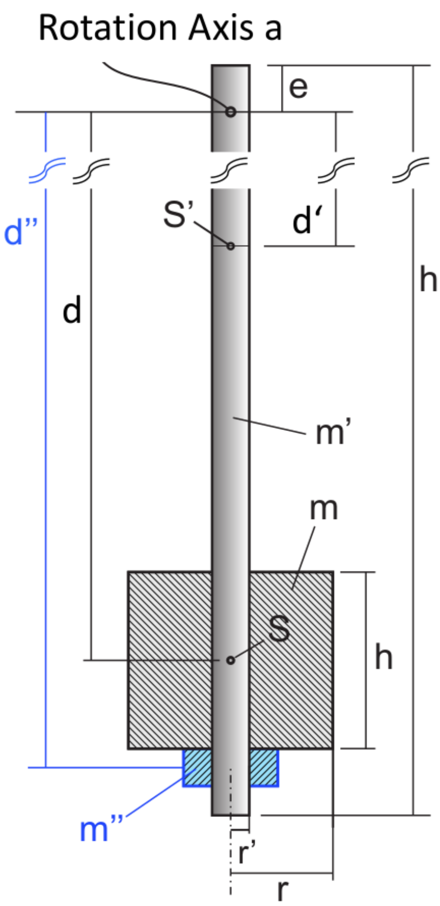

Figure 1: Schematic of one of the two identical pendulums used in the double pendulum. The two pendulums were connected by a spring connected to the shaft.

For each we measured the period of the oscillation, τ_ω, τ_Ω, and τ respectively again by measuring the time it took the pendulum to complete 50 oscillations. Then we calculated the theoretical values using equations (3), (4), and (5). Additionally for the wandering setup, we measured the time it took for one complete beating period to finish and then calculated the theoretical value using equation (6).

This whole experiment was conducted for two different degrees of coupling. The coupling was adjusted by changing the height of the spring between the pendulums.

Determining D_f and k Finally we determined the coupling moment D_f and the degree of coupling k for each of the couplings used in the previous experiment. We did this both dynamically, by using the results from the previous experiment and with equations (7) and (8), and statically, by measuring the displacement of one pendulum Φ_2 if the other displaced by a certain value Φ_1 and using equations (9) and (10).

Tracking the Pendulums In order to analyze the beating behavior of the double pendulum, we took an 80-second video of the pendulums swinging from a random starting position. We then tracked the tips of both pendulums and plotted their displacement against time to visualize the beating effect.

3 Results

The dimensions and masses of the parts in our setup are listed in table 1.

Shaft	Mass
$m' = (0.4300 \pm 0.0005)\,\mathrm{kg}$	$m = (1.0900 \pm 0.0005)\,\mathrm{kg}$
$h' = (0.7000 \pm 0.0005)\,\mathrm{m}$	$h = (0.0590 \pm 0.0005)\,\mathrm{m}$
$r' = (0.0270 \pm 0.0005)\,\mathrm{m}$	$r = (0.0500 \pm 0.0005)\,\mathrm{m}$

Table 1: The masses and dimensions of the two identical pendulums. The variables are named according to figure figure 1.

From these values we calculated the length of the pendulum to the center of mass l and its moment of inertia Θ:

$$
\begin{aligned}
l &= h' - \left(\frac{m}{m+m'}\frac{h}{2} + \frac{m'}{m+m'}\frac{h'}{2}\right) && \Longrightarrow l = (0.580 \pm 0.017)\,\mathrm{m} \\
\Theta &= \theta_M + m\left(h' - \frac{h}{2}\right)^2 + \theta_S + m'\left(\frac{h'}{2}\right)^2 && \Longrightarrow \Theta = (0.561 \pm 0.033)\,\mathrm{kgm}^2,
\end{aligned}
$$

where θ_M and θ_S are the moments of inertia of the center of mass of the mass and the shaft respectively. Here we calculated the error using Gauss error propagation.

Physical Pendulum The average time it took a single pendulum to oscillate $N = 50$ times was $T_{50} = (77.7 \pm 0.5)\,\text{s}$, where the error is the standard deviation from three trials. Therefore a single period is:

$$T_0 = (1.55 \pm 0.01)\,\text{s}.$$

Calculations for a mathematical and a physical pendulum yield the following results:

$$\begin{aligned} T_{0,\text{math}} &= 2\pi\sqrt{\frac{l}{g}} & \implies T_{0,\text{math}} &= (1.528 \pm 0.023)\,\text{s} \\ T_{0,\text{phys}} &= 2\pi\sqrt{\frac{\Theta}{(m+m')gl}} & \implies T_{0,\text{phys}} &= (1.600 \pm 0.024)\,\text{s}, \end{aligned}$$

where $g = 9.81$ is the gravitational acceleration on earth. For the errors, we used Gauss error propagation.

Coupled Oscillations Our measurements on the coupled oscillations can be found in table 2. The table shows the times for a single period. The times measured for 50 periods, can be found in the appendix. The theoretical values are calculated using equations (5) and (6).

Coupling 1	
measured	calculated
$\tau_\omega = (1.52 \pm 0.03)\,\text{s}$	
$\tau_\Omega = (1.46 \pm 0.01)\,\text{s}$	
$\tau = (1.520 \pm 0.005)\,\text{s}$	$\tau = (1.49 \pm 0.02)\,\text{s}$
$T_S = (22.1 \pm 0.4)\,\text{s}$	$T_S = (34 \pm 15)\,\text{s}$
Coupling 2	
measured	calculated
$\tau_\omega = (1.551 \pm 0.002)\,\text{s}$	
$\tau_\Omega = (1.517 \pm 0.002)\,\text{s}$	
$\tau = (1.520 \pm 0.002)\,\text{s}$	$\tau = (1.536 \pm 0.002)\,\text{s}$
$T_S = (64.7 \pm 0.4)\,\text{s}$	$T_S = (90 \pm 30)\,\text{s}$

Table 2: τ_ω is the period of the symmetric case, τ_Ω of the anti-symmetric case, τ of the wandering setup and T_S is the beating period. Measurements were taken of the time needed for 50 periods, but the values in the table show the time for a single period. The times for 50 periods can be found in the appendix. Their errors are the standard deviation from the three trials. The theoretical values are calculated according to (5) and (6). Their corresponding errors were calculated with Gauss error propagation.

Determining D_f and k Our measurements for Φ_1 and Φ_2 and the values calculated from (7), (8), (9) and (10) can be found in table 3.

Tracking the Pendulum Figure 2 shows the results of the video tracking. The beating effect between the two pendulums is very clear.

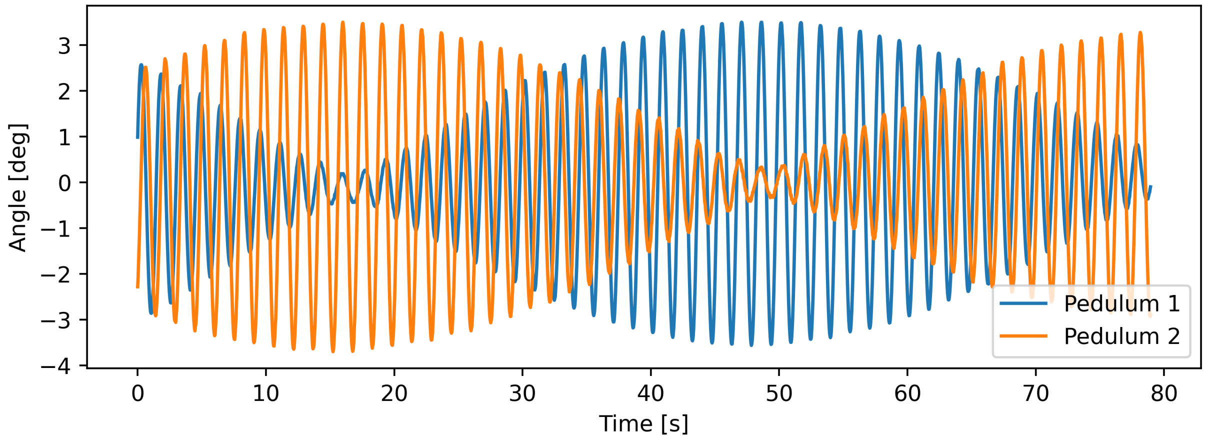

Figure 2: Angles of the two coupled pendulums over time. The beating effect can easily be seen.

Coupling 1	
Dynamic	Static
$D_f = (0.4 \pm 0.2)\,\mathrm{kgm^2/s^2}$ $k = 5 \pm 1$	$\Phi_1 = (8.0 \pm 0.1)^\circ$ $\Phi_2 = (0.5 \pm 0.1)^\circ$ $D_f = (0.5 \pm 0.1)\,\mathrm{kgm^2/s^2}$ $k = 6 \pm 1$
Coupling 2	
Dynamic	Static
$D_f = (0.17 \pm 0.06)\,\mathrm{kgm^2/s^2}$ $k = 1.9 \pm 0.6$	$\Phi_1 = (8.0 \pm 0.1)^\circ$ $\Phi_2 = (0.2 \pm 0.1)^\circ$ $D_f = (0.2 \pm 0.1)\,\mathrm{kgm^2/s^2}$ $k = 3 \pm 1$

Table 3: Here are the measured values for the coupling moment D_f and the degree of coupling k measured statically and dynamically, and the deflections measured $\Phi_1, |phi_2$ for the static method. The dynamic method's values were calculated with equations (7) and (8) and the values found in Table 2. The static method's values were calculated using equations (9) and (10). The errors for D_f and k were calculated using Gauss error propagation.

4 Discussion

The measured oscillation period for a single pendulum fell right in between the theoretical values for the mathematical and physical pendulum. For a more accurate measurement, we could have taken into account the screw, which held the mass on the shaft, into account when calculating the total moment of inertia and the position of the center of mass of the pendulum. We could have also taken the shaft off of the setup for a better measurement of the length.

While analyzing the coupled oscillations, we omitted one outlier from our results, a measurement of $\tau_\Omega = 1.54$s for the second coupling, where the other two values both were 1.52s. The reason for this outlier most likely is, that we miscounted the oscillations. Ignoring this data point yields the much more accurate calculation for $T_S = (90 \pm 30)\,\mathrm{s}$ as opposed to including it, which would have given us a beating period of $(100 \pm 200)\,\mathrm{s}$, which is a massive error range.

In order to obtain more accurate measurements for the oscillation period of a single pendulum T_0 and the oscillation period of a coupled pendulum moving symmetrically τ_ω and anti-symmetrically τ_Ω, we could conduct more trials, while reducing the number of total number of oscillations to for example $N = 10$, in order to minimize the possible errors from air resistance and friction, while also still trying to minimize the error from our reaction time.

The device we used to measure the deflection of the pendulum had units in increments of 0.5°, which made it difficult to get an accurate measurement used in calculating the coupling moment D_f and degree of coupling k in the static method. This is most likely the biggest factor in the difference in values between the static and dynamic methods. In order to measure better results, we could have used a tool with increments of for example 0.1°.

In a further investigation we could measure and analyze the effects of friction and air resistance by looking at how the period changes over time in an extended video analysis.

5 Conclusion

In conclusion we showed, that the period of our pendulum $T_0 = (1.55 \pm 0.01)\,\mathrm{s}$ lies between what a mathematical pendulum ($T_{0,\mathrm{math}} = (1.528 \pm 0.023)\,\mathrm{s}$) and a physical pendulum ($T_{0,\mathrm{phys}} = (1.600 \pm 0.024)\,\mathrm{s}$) predict.

Secondly we measured the periods of a symmetrically oscillating coupled pendulum and anti-symmetrically oscillating coupled pendulum for two separate couplings. The values are shown in table 2. From these we calculated the period for the case of a wandering oscillation, where one pendulum starts at a certain amplitude, while the other is in its null position. Further we calculated the period of the beating that results from such a setup. Both calculated values are compared to measured ones, as also can be seen in table 2.

Then we analyzed the two couplings by calculating D_f and k. We did this both Dynamically, using the values from the previous experiments, and statically, by measuring the static displacement of the two pendulums. The results are presented in table 3.

Finally we took a video of a coupled oscillation with an arbitrary starting position, and tracked the two pendulums. Their beating behavior can be seen in figure 2.

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6	**Gekoppelte Pendel**

1. Berechnung des Trägheitsmoments

Masse von Pendel und Regulierschraube	m = 1.090 ± 0.001 kg
Masse des Pendelschaftes	m'= 0.430 ± 0.0005 kg
Dimensionen der am Schaft befestigten Masse	h = 5.9±0.05cm r = 2.7±0.05cm
Dimensionen des Schaftes	h' = 70 ± 0.02cm r' = 0.5cm
Pendellänge (bis zum gemeinsamen Schwerpunkt von Pendelzylinder und Schaft):	ℓ = 0.58 ± 0.017m
Trägheitsmoment des Pendels berechnet	Θ= 0.561± 0.033

2. Physikalisches Pendel

Schwingungsdauer des physikalischen Pendels gemessen: 50 periods: T_{50} = 77.69s

Schwingungsdauer des mathematischen Pendels berechnet: T_o = 1.528 ± 0.023s

Abweichung in %:

Schwingungsdauer des physikalischen Pendels berechnet: T_o = 1.600 ± 0.029s

3. Gekoppelte Schwingungen:

	Kopplung I gemessen	Kopplung I berechnet	Kopplung II gemessen	Kopplung II berechnet
	50 oscillations		50 oscillations	
Symmetrische Schwingung	τ_ω = 76.05, 74.58, 77.7	XXXXXXXXX	τ_ω = 76.62, 77.62, 77.44	XXXXXXXXX
Antisymmetriche "	τ_Ω = 72.39, 72.51, 73.76	XXXXXXXXX	τ_Ω = 77.44, 75.95, 75.75	XXXXXXXXX
Wandernde "	τ = 72.35, 72.97, 72.9	τ = 1.49 ± 0.02 s	τ = 76.06, 75.9, 76.03	τ = 1.536 ± 0.002 s
Schwebungszeit	T_S = 22.59s, 22.10, 21.69	T_S = 34 ± 15 s	T_S = 64.65, 64.31, 65.19	T_S = 90 ± 30 s
Höhe der Federbefestigung	44 cm		25 cm	

4. Kopplung

	Kopplung I dynamisch	Kopplung I statisch	Kopplung II dynamisch	Kopplung II statisch
Kopplungsmoment	D_f = 0.4±0.2 kgm²/s²	Φ_1 = 8° Φ_2 = 0.5° D_f = 0.5±0.1 kgm²/s²	D_f = 0.17 ± 0.06 kgm²/s²	Φ_1 = 8° Φ_2 = 0.2° D_f = 0.2±0.1 kgm²/s
Kopplungsgrad in %	k = 5 ± 1	k = 6 ± 1	k = 1.9 ± 0.6	k = 3 ± 1

5. Zeitlicher Verlauf der Amplituden bei der Schwebung:

Grösse aufeinanderfolgender gleichsinniger Ausschläge:

Grafische Darstellung beider Pendel: A = f(t) [Auf Millimeterpapier]